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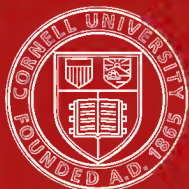
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DIFFERENTIAL AND INTEGRAL
CALCULUS.

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DIFFERENTIAL AND INTEGRAL CALCULUS,

WITH APPLICATIONS.

BY

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M.A., F.R.S.,

PROFESSOR OF MATHEMATICS TO THE SENIOR CLASS OF ARTILLERY OFFICERS, WOOLWICH.

SECOND EDITION.

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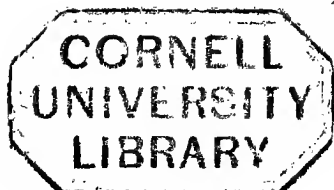


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PREFACE.

THE present Treatise is intended as an introduction to the study of the Differential and Integral Calculus, but will be found to contain what it is necessary to know in order to pass on to the subjects which presume a knowledge of the Calculus.

I have endeavoured to make this book suitable not only for the mathematical student, but also for men like engineers and electricians who require the subject for practical applications, to whom even a slight knowledge of the notation and methods of the Calculus is becoming more and more indispensable.

Hitherto in this country the influence of Newton, although the inventor of Fluxions, has been employed to delay the study of this subject and make a knowledge of it the privilege of a select few; my object in writing this treatise has been mainly to present the subject in as simple a manner as possible, in order to encourage a larger number of students to cultivate it.

With the object of keeping the size of the book within reasonable limits, it is assumed that the reader has already acquired a knowledge of the elements of Algebra, Trigonometry, and Coordinate Geometry, as given, for

instance, in the treatises of Hall and Knight, J. B. Lock and C. Smith; accordingly I have at once proceeded to the operation and application of Differentiation with as little preliminary explanation as possible.

I have followed the recent American treatises on this subject of Rice and Woolsey Johnson, Byerly and J. H. Taylor, in introducing the notion of Time as an independent variable, and the associated ideas of velocity and acceleration, in order to afford illustrations of the use of the Calculus; this is after all only a return to the Method of Fluxions as invented by Newton, and carried out by Maclaurin and other writers in this country, until supplanted by the notation of the Differential Coefficients of the foreign mathematicians.

The Doctrine of Fluxions is a useful and rigorous method of presenting the elementary ideas of the flow of varying quantities, and is employed in the treatises of Rice and Woolsey Johnson under the name of the Method of Rates; but the notation for a fluxion, for instance \dot{x} the fluxion of x , though easily written is difficult to print, and has the inconvenience of not indicating the independent variable, so that the notation of Leibnitz, $\frac{dx}{dt}$ instead of \dot{x} , is now used almost universally in printed books; and to economise space, this notation it is now proposed to print in the form dx/dt .

The chief novelties in the present work consist, first, in carrying on the subjects of the Differential and of the Integral Calculus together, instead of, as is usual, completing the Differential before passing on to the Integral Calculus; secondly, in the use of the hyperbolic functions in conjunction with the ordinary circular

trigonometrical functions, as thereby an exact analogy is preserved, which is not apparent when only the exponential and logarithmic functions are employed.

The notation of \sinh , \cosh , \tanh , etc., to denote the hyperbolic sine, cosine, tangent, etc., has been employed, in accordance with what appears to be now the most universal custom.

I have ventured also, on the grounds of symmetry, to introduce the inverse hyperbolic functions, and, following Ferrers and Byerly, to denote them by \sinh^{-1} , \cosh^{-1} , \tanh^{-1} , etc., by analogy with \sin^{-1} , \cos^{-1} , \tan^{-1} , etc.; this idea of symmetrical symbolism will be found indicated in Bertrand's *Integral Calculus*, Chapter I., but has not been pursued apparently because of the lengthiness of the notation there employed, namely, *sect. sin hyp.*, *sect. cos hyp.*, *sect. tang hyp.*, etc., instead of the above.

By the use of the direct and inverse hyperbolic functions in conjunction with the direct and inverse circular functions, the Calculus is in my opinion considerably simplified; and the student is led on more naturally and readily to the consideration of the elliptic and other functions. The consideration of these last functions is however beyond the scope of the present treatise.

To exhibit more clearly the analogy and symmetry between the circular and hyperbolic functions, I have made a digression in Chapter I. on the formulas of the addition equation (as it may be called by analogy with elliptic functions) of ordinary trigonometry, showing how the formulas may all be deduced from a single figure, with the corresponding relations of the hyperbolic functions.

Numerous collections of examples will be found throughout the book, introduced at each point to illustrate what has immediately gone before, and chosen as having some bearing on subsequent stages of mathematics.

The order of arrangement will be found in some respects different to what is customary; for instance, the idea of tracing simple curves from their equations has been introduced into the first chapter, so far as is required for the ordinary applications of the Integral Calculus to finding the areas, etc., of these curves; the general theory of curves being resumed in Chapter VI. Maximums and Minimums also have been investigated without the aid of Taylor's Theorem.

Change of the Independent Variable has only been touched upon where necessary; the general theory of Change of the Independent Variable, Lagrange's and Laplace's Theorems, and the Elimination of Constants and Functions, have been omitted as beyond the scope of an elementary practical treatise.

I have to thank Mr. A. G. Hadcock, Inspector of Ordnance Machinery, Royal Artillery, for drawing the diagrams, and also for revising the proof sheets and preparing the index.

WOOLWICH,
December, 1885.

PREFACE TO THE SECOND EDITION.

IN this second edition a rearrangement has been made, by which the general theory of Integration has been relegated to a supplementary final Chapter; only a slight preliminary sketch of Integration is now given in Chapter II., sufficient to carry on the student to some simple applications, and to enable him to follow the subsequent articles.

In this rearrangement, and in many other details at the outset, I have received valuable advice and assistance from Mr. W. Gallatly, M.A.

The theories of Change of the Variable, the Elimination of Constants and Functions, and the Theorems of Lagrange and Laplace have now been included, so that this new edition is increased in size.

By following the custom of giving the examples in smaller print, this increase in size might have been apparently much reduced; but it was decided to use a uniform type throughout the book, as less trying to the eyesight.

The *solidus* notation, for instance dx/dt for $\frac{dx}{dt}$, has now been used sparingly, in places where no confusion

or ambiguity could arise, and thereby some economy of space has been obtained.

A few pages of explanation of the simplest Differential Equations which occur in Dynamics and Electricity have been added; while the illustrations of Change of the Variable have been chosen from among the most typical differential equations; and it is hoped that this slight sketch will enable the student to solve an ordinary differential equation which may come in his way.

The greater part of the diagrams have been redrawn by Mr. A. G. Hadcock, who has again helped me in the revision of the proof sheets and the preparation of the Index.

WOOLWICH,

March, 1891.

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Articles in the earlier chapters marked with a * may be omitted at a first reading.

ERRATA.

Page 82, ex. 39, *read*—"with $X = ax^4 + 4bx^3 + 6cx^2 + 4b'x + a'$."

Page 125, line 7, *read*—" $\frac{4}{3}\pi y^3 = \frac{1}{3}\pi \times PP'^3$; "

Page 139, ex. 8, *read*—" $\frac{d^n y}{dx^n} = \frac{(n-1)! \{ (1+x)^n - (-1+x)^n \}}{(1-x^2)^n}$.,"

Page 142, last line but one to be cancelled.

Page 143, ex. 2. iv., *read*—" $y^{(2n+1)} = (-1)^n 2n! \alpha^{-2n-1}$."

Page 151, ex. 18, the result as given is incorrect.

Page 160, line 3 from bottom, *read*—" $\frac{1}{2}(V+u)$."

Page 183, line 15, *read*—" $Cu = u_1 \int S u_3 dx - u_2 \int S u_1 dx$."

Page 231, line 4 from bottom, *read*—"partial fractions."

DIFFERENTIAL AND INTEGRAL CALCULUS.

CHAPTER I.

DIFFERENTIATION.

1. *Introduction. Constant and Variable Quantities.*

The Calculus to be developed in the present treatise is the method of reasoning applicable to variable quantities in a state of continuous change.

We call quantities *variable* when they change gradually by increase or decrease; on the contrary *constant*, when they remain unchanged while others change.

Thus the abscissa and ordinate of points on a parabola vary, while the parameter remains constant.

Again, the distance of a railway station from the terminus, or the latitude and longitude of a rock, of a lightship, or of an observatory, are constant quantities.

But the distance of a railway train from the terminus, or the latitude and longitude of a steamer or traveller are variable quantities.

Constant quantities are generally represented algebraically by the first letters of the alphabet, such as a, b, c, \dots , or A, B, C, \dots , or $\alpha, \beta, \gamma, \dots$; while variable quantities are represented by the last letters, $\dots t, u, v, w, x, y, z$, or $\dots X, Y, Z$, or $\dots \xi, \eta, \zeta$.

2. Definition of a Function; and of Independent and Dependent Variables.

One variable quantity, denoted by y or $f(x)$, is said to be a *function* of another variable quantity, denoted by x , when the value of y or $f(x)$ depends on the value of x .

The variable x , called the *independent variable* or *argument*, is one to which any value may be arbitrarily assigned.

The variable y or $f(x)$, called the *dependent variable*, is one of which the value is determined, when the arbitrary value of the independent variable x has been assigned.

We may use the notation fx instead of $f(x)$ when the independent variable consists of a single term like x .

Thus $x^2, x^3, x^4, \dots, x^n, \sqrt{x}, \sqrt[n]{x}, \sin x, \cos x, \tan x, \cot x, \sec x, \operatorname{cosec} x, \operatorname{vers} x, \sin^{-1}x, \cos^{-1}x, \tan^{-1}x, \cot^{-1}x, \sec^{-1}x, \operatorname{cosec}^{-1}x, \operatorname{vers}^{-1}x, a^x, \exp x, \cosh x, \sinh x, \tanh x, \dots \log x, \cosh^{-1}x, \sinh^{-1}x, \tanh^{-1}x, \dots$, all denoted generically by fx , and most of them presumably familiar to the student, are simple functions of x , which we shall require hereafter.

We proceed to *differentiate* them, that is to find the *differential coefficient*, which is defined as follows:—

3. Definition of a Differential Coefficient.

If fx denotes any function of a variable quantity x , and if $f(x+h)$ denotes the *same* function of $x+h$ when x receives a small *increment* h , then the *limiting value* of

$$\frac{f(x+h) - fx}{h},$$

when h is indefinitely diminished, is called the *differential coefficient* of fx with respect to x , and is denoted by

$$\frac{dfx}{dx} \text{ or } f'x.$$

This definition may be conveniently expressed as

$$\frac{dfx}{dx} = \text{lt} \frac{f(x+h) - fx}{h},$$

(lt) being the abbreviation employed to denote the limiting value, as h is indefinitely diminished and ultimately becomes zero.

Since $f(x+h) - fx$ is the increment of fx corresponding to the increment h of x , therefore

$$\text{lt} \frac{f(x+h) - fx}{h}$$

is the *ultimate ratio* of the corresponding increments of fx and x , denoted by dfx and dx , and called the *differentials* of fx and x ; and thus $\frac{dfx}{dx}$ measures the *rate of increase or growth* of fx ; while $\frac{f(x+h) - fx}{h}$ represents the *average rate* of increase of fx from x to $x+h$.

The chief object at the outset of our subject is the determination of the differential coefficients of functions, and the application of them to the discussion of the geometrical and analytical properties of the functions.

The algebraical difficulty in the determination of the differential coefficient lies in the reduction of its original *indeterminate form* $\frac{0}{0}$ to a determinate limit.

(Hall and Knight, *Higher Algebra*, chap. xx.)

The name *derivative* or *derived function* is sometimes used instead of *differential coefficient*.

The differential coefficient $f'x$ of a function of fx may be supposed to derive its name from being the coefficient which turns the differential of x into the differential of fx .

(Wicksteed, *The Alphabet of Economic Science*, p. 32.)

4. *Differential Coefficient or Derivative of x^n .*

From the definition of a differential coefficient in § 3,

$$\begin{aligned}\frac{dx}{dx} &= \lim_{h \rightarrow 0} \frac{x+h-x}{h} = 1. \\ \frac{dx^2}{dx} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \frac{0}{0} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \frac{0}{0} \\ &= \lim_{h \rightarrow 0} (2x + h) = 2x. \\ \frac{dx^3}{dx} &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \frac{0}{0} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \frac{0}{0} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2. \\ \frac{dx^4}{dx} &= \lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h} = \frac{0}{0} \\ &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \frac{0}{0} \\ &= \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3.\end{aligned}$$

And generally, when n is a positive integer,

$$\frac{dx^n}{dx} = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} = \frac{0}{0}$$

(and expanding by the Binomial Theorem)

$$\begin{aligned}&= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \frac{n(n-1)}{1 \cdot 2}x^{n-2}h^2 + \dots + h^n}{h} = \frac{0}{0} \\ &= \lim_{h \rightarrow 0} \left\{ nx^{n-1} + \frac{n(n-1)}{1 \cdot 2}x^{n-2}h + \dots + h^{n-1} \right\} = nx^{n-1}.\end{aligned}$$

By assuming the *convergency* of the Binomial Theorem we can make the same proof hold for showing that the differential coefficient of x^n with respect to x is nx^{n-1} ,

where n represents any constant number, whether integral, fractional, positive or negative.

But without using the Binomial Theorem or assuming its convergency, a proof may be given as follows:—

(i.) Suppose n a positive integer, and denote $x+h$ by x_1 ; then

$$\frac{dx^n}{dx} = \lim_{x_1 \rightarrow x} \frac{x_1^n - x^n}{x_1 - x} = \frac{0}{0}$$

(and then by Division)

$$= \lim_{x_1 \rightarrow x} (x_1^{n-1} + x x_1^{n-2} + \dots + x^{n-2} x_1 + x^{n-1}) = n x^{n-1},$$

when $h=0$, and therefore $x_1=x$.

(ii.) Suppose n a positive fraction p/q , where p and q are positive integers; and put $x=z^q$, $x_1=z_1^q$; then

$$\begin{aligned} \frac{dx^n}{dx} &= \lim_{x_1 \rightarrow x} \frac{x_1^{p/q} - x^{p/q}}{x_1 - x} \\ &= \lim_{z_1 \rightarrow z} \frac{z_1^p - z^p}{z_1^q - z^q} \end{aligned}$$

(dividing out $z_1 - z$ from numerator and denominator)

$$\begin{aligned} &= \lim_{z_1 \rightarrow z} \frac{z_1^{p-1} + z_1^{p-2} z + \dots + z_1 z^{p-2} + z^{p-1}}{z_1^{q-1} + z_1^{q-2} z + \dots + z_1 z^{q-2} + z^{q-1}} \\ &= \frac{p z^{p-1}}{q z^{q-1}} = \frac{p}{q} z^{p-q} = n x^{n-1}. \end{aligned}$$

(iii.) Suppose n any negative number $-m$; then

$$\begin{aligned} \frac{dx^n}{dx} &= \lim_{x_1 \rightarrow x} \frac{x_1^{-m} - x^{-m}}{x_1 - x} = - \lim_{x_1 \rightarrow x} \frac{x_1^m - x^m}{x_1 - x} \frac{1}{x_1^m x^m} \\ &= - \frac{m x^{m-1}}{x^{2m}} = -m x^{-m-1} = n x^{n-1}; \end{aligned}$$

so that $\frac{dx^n}{dx} = n x^{n-1}$, universally, where n denotes any constant quantity.

Examples.—1. Determine from the definition the d.c. (differential coefficient) with respect to x of

$$\sqrt{x}, x^{\frac{1}{3}}, x^{\frac{2}{3}}, x^{\frac{1}{4}}, x^{\frac{3}{4}}, x^{\frac{5}{4}}, \frac{1}{\sqrt{x}}, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^n}, (x+a)^n,$$

$$\left(\frac{x}{n}\right)^n, \frac{2x+3}{x+1}, \frac{ax+b}{Ax+B}, a+bx+cx^2+\dots$$

2. Give a rigorous proof that the differential coefficient of $x^{\sqrt{2}}$ with respect to x is $\sqrt{2} x^{\sqrt{2}-1}$.

5. *Geometrical Interpretation of Differentiation.*

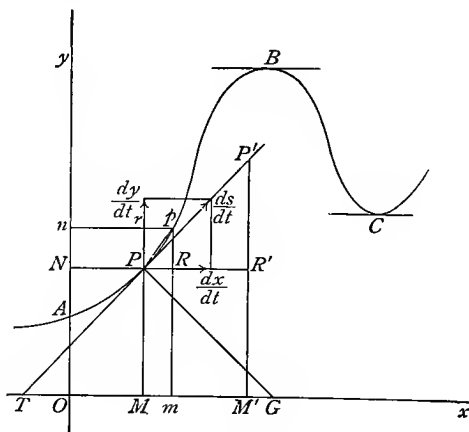


Fig.1

In Figure 1 the coordinates of a point P , referred to axes Ox and Oy at right angles, are denoted by x and y , where $OM=x$, $MP=y$; and in general, in the vertical elevation of an object, x is measured positive to the right, and negative to the left, y is measured positive upwards, and negative downwards; or, on a map or plan, Ox is drawn to the east, and Oy to the north.

Then the equation $y = fx$ represents some curve APB , the assemblage of points whose coordinates satisfy this equation; so that if $OM = x$, then $MP = fx$; the curve APB is now called the *graph* of the function fx (Chrystal, Algebra I., chap. xv.).

If $Mm = h$, then $Om = x + h$,

and $mp = f(x + h)$, $Rp = f(x + h) - fx$;

so that
$$\frac{f(x + h) - fx}{h} = \frac{Rp}{PR} = \tan RPp.$$

Now if the angle xTP is denoted by ψ , then since the direction of the tangent TP is the ultimate direction of the chord Pp when the point p has approached indefinitely near to P , therefore

$$\begin{aligned}\tan \psi &= \text{lt } \tan RPp = \text{lt } \frac{Rp}{PR} \\ &= \text{lt } \frac{f(x + h) - fx}{h} = \frac{dfx}{dx}, \text{ or } f'x, \text{ or } \frac{dy}{dx}, \text{ or } dy/dx,\end{aligned}$$

since $y = fx$.

By giving x a *decrement* h as well as an increment h , we may define a differential coefficient by the relation

$$\frac{dfx}{dx} = \text{lt } \frac{f(x + h) - f(x - h)}{2h},$$

which is sometimes more elegant and useful than the definition in § 3, to which it is ultimately equivalent.

Figure 1 shows that dy/dx or $\tan \psi$ is positive if y increases with x , but that dy/dx is negative if y diminishes as x increases; and thus dy/dx is zero at the *turning points*, such as B or C , where y from increasing begins to diminish, or from diminishing begins to increase.

At B , y is said to have a *maximum* value, and at C a *minimum* value; and in each case $dy/dx=0$.

Thus to discover the maximum or minimum value of y , a function of x , we must first find the values of x which make $dy/dx=0$.

Afterwards we must determine whether, as x increases continually, dy/dx changes from positive to negative, in which case y has a *maximum* value; or whether dy/dx changes from negative to positive, in which case y has a *minimum* value; we shall return to this subject hereafter, although the present application to simple cases is easy.

Thus $ax-x^2$ has a maximum value $\frac{1}{4}a^2$, when $x=\frac{1}{2}a$ (fig. 33); and $2x^3-9x^2+12x-3$ has a maximum value 2 when $x=1$, and a minimum value 1 when $x=2$ (fig. 29).

Imagine the curve ABC to be a road, and dx a small horizontal forward step, dy the consequent vertical step; then dy and therefore dy/dx is positive if the traveller is ascending, and dy and dy/dx is negative if the traveller is descending, supposing dx always positive.

At the turning points, such as the top of a hill or the bottom of a valley, where y is a maximum or minimum, $dy/dx=0$.

The road is steepest where dy/dx has its maximum numerical value, and then the differential coefficient of dy/dx will be zero.

We may call $\tan \psi$ or dy/dx the *gradient* of the road, the ascent being dy/dx feet vertical for one foot horizontal; or otherwise expressed, as on railway plans, the gradient is 1 in $\cot \psi$ or dx/dy , meaning 1 foot vertical for dx/dy feet horizontal.

6. *The Tangent and Normal of a Curve.*

If x', y' are the coordinates of any point P' on the tangent TPP' to the curve APB at P (fig. 1), then

$$\frac{y' - y}{x' - x} = \frac{dy}{dx}, \quad \text{or} \quad y' - y = \frac{dy}{dx}(x' - x),$$

the equation of the tangent TP .

TM is called the *subtangent*, and MG the *subnormal* at P ; PG being the *normal* at P , drawn perpendicular to the tangent through P . Therefore

$$TM = y \cot \psi = y dx/dy; \quad MG = y \tan \psi = y dy/dx.$$

If T is to the right of M , $\tan \psi$ is negative, and

$$MT = -y dx/dy, \quad GM = -y dy/dx.$$

Also the equation of the normal GP at P is

$$y' - y = -\frac{dx}{dy}(x' - x).$$

Let us apply these principles to some simple curves, the graphs of (i.) $y = x^2$, (ii.) $y = x^{\frac{1}{2}}$, (iii.) $y = x^{-1}$, (iv.) $y^2 = x^3$.

(i.) $y = x^2$, a *parabola* (fig. 2 i.).

Here $dy/dx = 2x$; so that the equation of the tangent is $y' - y = 2x(x' - x)$; which, since $y = x^2$, may be written

$$2xx' = y' + y, \text{ or } \frac{x'}{\frac{1}{2}x} - \frac{y'}{y} = 1.$$

Thus $OT = \frac{1}{2}x$, $TM = \frac{1}{2}x$, and therefore T bisects OM .

The equation of the normal is $\frac{x'}{2x^3 + x} + \frac{y'}{y + \frac{1}{2}} = 1$.

(ii.) $y = x^{\frac{1}{2}}$, another *parabola* (fig. 2 ii.).

Here $dy/dx = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}y/x$;

so that the equations of the tangent and normal become

$$\frac{y'}{\frac{1}{2}y} - \frac{x'}{x} = 1; \quad \frac{x'}{x + \frac{1}{2}} + \frac{y'}{2y^3 + y} = 1;$$

thus $TO = OM$, $OV = VN$, and $MG = \frac{1}{2}$.

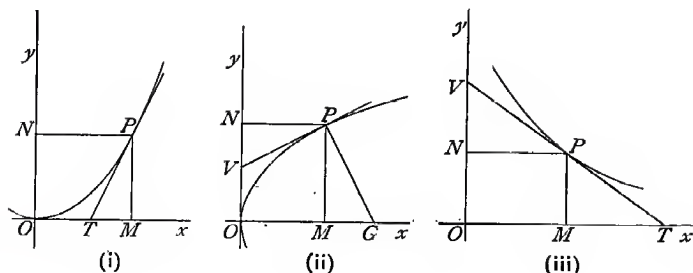


Fig. 2

(iii.) $y = x^{-1}$, a *hyperbola* (fig. 2 iii.).

Here
$$\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2} = -\frac{y}{x};$$

and the equation of the tangent becomes

$$\frac{x'}{2x} + \frac{y'}{2y} = 1,$$

so that

$$OT = 2x, OV = 2y.$$

The equation of the normal is

$$xx' - yy' = x^2 - y^2.$$

(iv.) $y^2 = x^3$, or $y = x^{\frac{3}{2}}$, a curve called the *semicubical parabola*, and something like figure (i.).

Here $dy/dx = \frac{3}{2}x^{\frac{1}{2}}$, so that $TM = \frac{2}{3}x$, $OT' = \frac{1}{3}x = \frac{1}{3}OM$.

(v.) Prove that in the curve $y^m = cx^n$, the equations of the tangent and normal are

$$\frac{x'}{mx} - \frac{y'}{ny} = \frac{1}{m} - \frac{1}{n}, \quad mxx' + nyy' = mx^2 + ny^2.$$

* 7. Supposing h again, as at first, to denote a finite increment of x , and supposing that neither fx nor $f'x$ becomes infinite between P and p , and that they change gradually between these points; then in the graph of fx

* Articles which may be omitted at a first reading are marked with an asterisk *

the tangent TK at some point K between P and p (fig. 3) is parallel to the chord Pp ; and

$$\frac{f(x+h) - fx}{h} = \tan RPp = \tan xTK = f'(x + \theta h),$$

where $x + \theta h = OH$, the abscissa of K ; and then $\theta = MH/Mm$, a proper fraction, which is some unknown function of x and h .

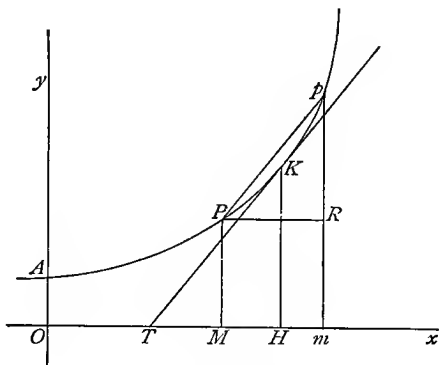


Fig. 3

Therefore $f(x+h) = fx + hf'(x + \theta h)$,
a theorem required subsequently in Chapter IV.

This leads to another definition of $f'x$, employed by Lagrange, according to which $f'x$ is defined as the coefficient of h in the algebraical expansion of $f(x+h)$ in ascending positive integral powers of h .

To illustrate geometrically exceptional cases of fx and $f'x$, consider $fx = (x-b)/(x-a)$, or $fx = (x-a)^{\frac{2}{3}}$, taking $x < a$, and $x+h > a$.

Given that $fx = a + bx + cx^2$,
then $\theta = \frac{1}{2}$; and generally, when h is small, we shall find $\theta \approx \frac{1}{2}$; the symbol \approx denoting approximate equality.

If k denotes a different increment of x , then

$$f(x+k) = fx + kf'(x+\phi k),$$

where ϕ is a proper fraction, the same function of x and k which θ is of x and h ; so that

$$\frac{f(x+h) - fx}{f(x+k) - fx} = \frac{h}{k} \frac{f'(x+\theta h)}{f'(x+\phi k)}.$$

Ultimately, when h and k are sufficiently small, we may put

$$\frac{f'(x+\theta h)}{f'(x+\phi k)} = 1,$$

without sensible error; and then

$$\frac{f(x+h) - fx}{f(x+k) - fx} = \frac{h}{k},$$

which is the rule of *proportional parts*, employed in Trigonometry; equivalent to neglecting the curvature of the arc Pp , and replacing the arc by a straight line.

Again supposing APp to represent the profile or elevation of a road, and $f'x$ to represent the slope or gradient at P , $f'(x+h)$ at p , then the gradient $f'(x+\theta h)$ at K is the average gradient of the chord Pp , represented by $\{f(x+h) - fx\}/h$.

8. Instead of the letter h the symbol Δx is often employed to denote the finite increment of x ; and Δy is then used to denote the corresponding increment of y , where y is a function of x , denoted by fx .

Then, with this notation,

$$y + \Delta y = f(x + \Delta x),$$

and $\Delta y = f(x + \Delta x) - fx$;

so that $\Delta y = R p$, if $\Delta x = PR$ (fig. 1);

and $\frac{dy}{dx} = \lim_{\Delta x} \frac{\Delta y}{\Delta x}.$

In this notation, Δ (Delta) and d are not algebraical factors, but symbolic operations; so that Δx and dx must be considered as inseparable quantities, the x being qualified by the Δ or d , and not multiplied algebraically by it.

Suppose for instance it is found that the difference of range of a gun over the sea at high and at low water is Δx , when Δy represents the rise and fall of the tide; then the tangent of the angle of descent of the projectile into the water is $\Delta y/\Delta x$, approximately.

9. Again, let the length of the arc AP of the curve APp , measured from any fixed point A to the variable point P , be denoted by s (fig. 1).

Denoting by Δs the increment of s , corresponding to the increment Δx of x , then the arc $APp = s + \Delta s$, and the arc $Pp = \Delta s$.

Now, when the point p approaches to coincidence with P along the curve pP , it is assumed as axiomatic that

$$\lim_{p \rightarrow P} \frac{\text{chord } Pp}{\text{arc } Pp} = 1;$$

$$\text{or} \quad \lim_{p \rightarrow P} \frac{\sqrt{(PR^2 + R^2 p^2)}}{\text{arc } Pp} = 1;$$

$$\text{or} \quad \lim_{p \rightarrow P} \frac{\sqrt{(\Delta x^2 + \Delta y^2)}}{\Delta s} = 1;$$

where Δx^2 means $(\Delta x)^2$;

$$\text{or} \quad \lim_{p \rightarrow P} \left(\frac{\Delta x^2}{\Delta s^2} + \frac{\Delta y^2}{\Delta s^2} \right) = 1.$$

$$\text{Therefore} \quad \frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} = 1,$$

or, as it is sometimes written,

$$\left(\frac{dx}{ds} \right)^2 + \left(\frac{dy}{ds} \right)^2 = 1.$$

The assumption, which has been taken as axiomatic, that $\text{lt}(\text{chord } Pp)/(\text{arc } Pp) = 1$, is proved in Newton's Lemma VII. (*Principia*, Book I, § 1), as follows—

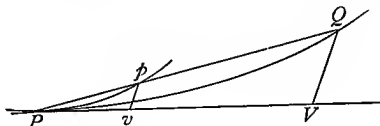


Fig. 4

As the point p approaches to the point P , let the chord Pp , and the tangent Pv be produced to points Q , V , at a finite distance, where QV is parallel to pv , and the arc PQ is similar to the arc Pp ; so that PQV may be considered the magnified image of the similar microscopic figure Ppv .

Now let the point p move on the curve pP close up to P , and let pv move parallel to itself up to P ; then the angle QPv always diminishes and ultimately vanishes; and the chord PQ , the arc PQ and the tangent PV are ultimately coincident; therefore also the chord Pp , the arc Pp and the tangent Pv , when vanishing together, will have a ratio of equality.

$$\begin{aligned} \text{Again (fig. 1) } \cos \psi &= \cos xTP = \text{lt } \cos RPp = \text{lt } \frac{PR}{Pp} \\ &= \text{lt } \frac{\Delta x}{\Delta s} \text{lt } \frac{\text{arc } Pp}{\text{chord } Pp} = \frac{dx}{ds}; \end{aligned}$$

$$\begin{aligned} \text{and } \sin \psi &= \sin xTP = \text{lt } \sin RPp = \text{lt } \frac{RP}{Pp} \\ &= \text{lt } \frac{\Delta y}{\Delta s} \text{lt } \frac{\text{arc } Pp}{\text{chord } Pp} = \frac{dy}{ds}. \end{aligned}$$

$$\text{Since } (\cos \psi)^2 + (\sin \psi)^2 = 1,$$

$$\text{therefore } \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1, \text{ as before.}$$

The tangent $TP = y \operatorname{cosec} \psi = y ds/dy$; and the normal $PG = y \sec \psi = y ds/dx$; and the length of the perpendicular p from the origin O upon the tangent at P is

$$p = x \sin \psi - y \cos \psi = x \frac{dy}{ds} - y \frac{dx}{ds}.$$

Since $(\sec \psi)^2 = 1 + (\tan \psi)^2,$

and $\sec \psi = ds/dx, \tan \psi = dy/dx;$

therefore $\frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2}.$

Similarly, from $(\operatorname{cosec} \psi)^2 = (\cot \psi)^2 + 1,$

we deduce $\frac{ds^2}{dy^2} = \frac{dx^2}{dy^2} + 1.$

Sometimes the gradient is measured by the vertical rise for one foot on the incline, that is by dy/ds or the sine of the slope ψ ; and then the gradient is one in ds/dy or $\operatorname{cosec} \psi$.

We may say that the gradient is measured by the tangent of the slope on the plan, and by the sine of the slope on the constructed work; in the inclines of roads and railways the slope is sufficiently small for no difference to be perceptible in the two methods.

10. *Velocity expressed by a Differential Coefficient.*

If x and y , the coordinates of a point P , moving along the curve APp (fig. 1); the path of a projectile in the air or of a train on a railway, for instance) are given as functions of the time t ; and if Δt denotes the time occupied in moving from P to p , a distance Δs ; then the *average* velocity from P to p is $\Delta s/\Delta t$; and the actual velocity of P , in the direction of the tangent TP , is

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt},$$

the rate of increase or growth of s per unit of t .

The component velocity of P parallel to Ox is the velocity of M ; and treating M as a point moving in the straight line Ox , the *average* velocity from M to m is $\Delta x/\Delta t$; and therefore the actual velocity of M

$$= \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt},$$

the rate of growth of x per unit of t .

Similarly the component velocity of P parallel to Oy , or the rate of growth of y is dy/dt , the velocity of N .

Since the resultant velocity of P , ds/dt , has the components dx/dt and dy/dt in two rectangular directions, therefore

$$\frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}.$$

Making $t = s, x$, or y , we obtain the previous results.

According to Newton's original definition, the *fluxion* of a variable quantity x is its velocity or rate of growth, and was denoted by \dot{x} ; in Leibnitz's notation of the Differential Calculus it is represented by $\frac{dx}{dt}$ or dx/dt ;

thus differentiation with respect to the time t may be considered the typical fundamental conception of this Calculus, which investigates the rate at which variable quantities change, now not only with respect to the time t , but also with respect to any other independent variable.

11. Differentiation of a Function of a Function.

Since
$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \frac{\Delta x}{\Delta t},$$

therefore, proceeding to the limit,

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt},$$

where x is a function of t , and y is a function of x , and therefore of t .

Thus, with the notation of §§ 9, 10, (fig. 1),

$$\frac{dx}{dt} = \frac{dx}{ds} \frac{ds}{dt} = \cos \psi \frac{ds}{dt}, \quad \frac{dy}{dt} = \frac{dy}{ds} \frac{ds}{dt} = \sin \psi \frac{ds}{dt};$$

showing how the component velocities are obtained by resolving the resultant velocity.

Denoting the velocity ds/dt by v , then dv/dt , the rate of growth of v , is called the *acceleration*; and

$$\frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt} = \frac{dv}{ds} v = \frac{d\frac{1}{2}v^2}{ds}.$$

With different letters, supposing z is a function of y , and y is a function of the independent variable x ; and supposing the increment Δx of x gives y the increment Δy , and z the increment Δz ; then since, algebraically,

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y} \frac{\Delta y}{\Delta x},$$

therefore
$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx},$$

the rule for the *differentiation of a function of a function*.

Thus, suppose $z = y^m$, where y is a function of x ; then

$$\frac{dz}{dx} = \frac{dy^m}{dx} = \frac{dy^m}{dy} \frac{dy}{dx} = m y^{m-1} \frac{dy}{dx};$$

or, if $y = fx$, then
$$\frac{d(fx)^m}{dx} = m(fx)^{m-1} f'x.$$

Putting $z = Fy$ and $y = fx$, so that $z = F(fx)$; then

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = F'y f'x = F'(fx) f'x.$$

Thus if $z = (a + bx^n)^m$,
$$\frac{dz}{dx} = m(a + bx^n)^{m-1} n b x^{n-1}.$$

With $y = fx$, then $dy/dt = f'x \cdot dx/dt$; and this, in the notation of *Differentials*, is abbreviated to $dy = f'x \cdot dx$, in which the independent variable is left arbitrary; and dx , dy are called the differentials of x and y , as in § 3.

12. Differentiation of the Sum, Product, and Quotient of Functions.

Let u and v denote any two functions of x ; and let Δu and Δv denote the increments of u and v , corresponding to the increment Δx of x ; then (i.) for their sum,

$$\frac{d(u+v)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta u + \Delta v}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \right) = \frac{du}{dx} + \frac{dv}{dx};$$

so that the *d.c.* of the sum of two functions is the sum of the *d.c.*'s of the functions.

Similarly, for the difference of u and v ,

$$\frac{d(u-v)}{dx} = \frac{du}{dx} - \frac{dv}{dx}.$$

And generally, if a and b denote constant factors,

$$\frac{d(au+bv)}{dx} = a \frac{du}{dx} + b \frac{dv}{dx}.$$

$$\begin{aligned} \text{Thus } \frac{d}{dx}(ax^m + bx^n - cx^p + \dots) \\ = amx^{m-1} + bnx^{n-1} - cpx^{p-1} + \dots; \end{aligned}$$

and if $s = ut + \frac{1}{2}ft^2$, $v = ds/dt = u + ft$.

We notice now that the differential coefficient of a *constant* is zero; as is obvious, since a constant does not change with the independent variable.

(ii.) For the product of u and v ,

$$\begin{aligned} \frac{d(uv)}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{(u + \Delta u)(v + \Delta v) - uv}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} v + \lim_{\Delta x \rightarrow 0} (u + \Delta u) \frac{\Delta v}{\Delta x} = \frac{du}{dx} v + u \frac{dv}{dx}; \end{aligned}$$

because $\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx}$, $\lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} = \frac{dv}{dx}$, and $\lim_{\Delta x \rightarrow 0} (u + \Delta u) = u$.

Hence the rule for differentiating a product: *Differentiate with respect to each factor and add the results.*

For instance if $u = (x-a)^m$, $v = (x-b)^n$,

$$\begin{aligned}\frac{d(x-a)^m(x-b)^n}{dx} &= m(x-a)^{m-1}(x-b)^n + (x-a)^m n(x-b)^{n-1} \\ &= (x-a)^{m-1}(x-b)^{n-1} \{ (m+n)x - mb - na \}.\end{aligned}$$

It is important for subsequent purposes not to alter the order of the factors in differentiating; thus for a product of three factors u, v, w ,

$$\frac{duvw}{dx} = \frac{du}{dx}vw + u\frac{dvw}{dx} = \frac{du}{dx}vw + u\frac{dv}{dx}w + uv\frac{dw}{dx};$$

and so on for any number of factors.

To illustrate geometrically, refer to fig. 1, in which P is a point moving along the curve AP , whose co-ordinates are x, y at the time t .

In the time Δt , P has moved to p , whose coordinates are $x + \Delta x, y + \Delta y$; and the rectangle $OMPN = xy$ has increased by the rectangles $Pm = \Delta x.y$, and $nR = (x + \Delta x)\Delta y$; and the formula

$$\frac{dxy}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}$$

expresses the rate of growth of the rectangle xy in the two directions Ox and Oy .

(iii.) For u/v , the quotient of u by v ,

$$\begin{aligned}\frac{d\frac{u}{v}}{dx} &= \lim_{\Delta x} \frac{\frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}}{\Delta x} = \lim_{\Delta x} \frac{(u + \Delta u)v - u(v + \Delta v)}{\Delta x(v + \Delta v)v} \\ &= \lim_{\Delta x} \frac{\frac{\Delta u}{\Delta x}v - u\frac{\Delta v}{\Delta x}}{(v + v\Delta)v} = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2};\end{aligned}$$

hence the rule for differentiating a fraction: *the differential coefficient is equal to*

$$\frac{d.c. \text{ of } num^r. \times denom^r. - num^r. \times d.c. \text{ of } denom^r.}{(denominator)^2}.$$

Thus if $y = \frac{3x - x^3}{1 - 3x^2}$

$$\frac{dy}{dx} = \frac{(3 - 3x^2)(1 - 3x^2) - (3x - x^3)(-6x)}{(1 - 3x^2)^2} = 3\left(\frac{1 + x^2}{1 - 3x^2}\right)^2.$$

If the denominator of the fraction is of the form v^n , for instance, if $y = u/v^n$,

then
$$\frac{dy}{dx} = \frac{\frac{du}{dx}v^n - unv^{n-1}\frac{dv}{dx}}{v^{2n}} = \frac{\frac{du}{dx}v - nu\frac{dv}{dx}}{v^{n+1}},$$

after removing the common factor v^{n-1} from the numerator and denominator.

Thus if $y = \frac{(x-a)^m}{(x-b)^n}$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{m(x-a)^{m-1}(x-b)^n - (x-a)^mn(x-b)^{n-1}}{(x-b)^{2n}} \\ &= \frac{(x-a)^{m-1}\{(m-n)x - mb + na\}}{(x-b)^{n+1}}.\end{aligned}$$

In such cases it is sometimes preferable to write the fraction in the form of a product; then $y = uv^{-n}$;

and
$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx}v^{-n} - unv^{-n-1}\frac{dv}{dx} \\ &= \frac{\frac{du}{dx}v - nu\frac{dv}{dx}}{v^{n+1}},\end{aligned}$$

as before, but without introducing extraneous factors.

Thus if $y = \frac{1}{(x-a)^m(x-b)^n}$,

writing it in the form

$$y = (x-a)^{-m}(x-b)^{-n},$$

then
$$\begin{aligned}\frac{dy}{dx} &= -m(x-a)^{-m-1}(x-b)^{-n} - (x-a)^{-m}n(x-b)^{-n-1} \\ &= -\frac{(m+n)x - na - mb}{(x-a)^{m+1}(x-b)^{n+1}}.\end{aligned}$$

Examples of Differentiation.

(1) Prove the following differentiations

$$(i.) \quad y = \sqrt{(2x)}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{(2x)}}.$$

(Solution.—Here $y = (2x)^{\frac{1}{2}}$, and therefore

$$\frac{dy}{dx} = \frac{1}{2} \times (2x)^{-\frac{1}{2}} \times 2 = (2x)^{-\frac{1}{2}} = \frac{1}{\sqrt{(2x)}}.)$$

$$(ii.) \quad y = (nx)^{\frac{1}{n}}, \quad \frac{dy}{dx} = (nx)^{\frac{1}{n}-1}.$$

$$(iii.) \quad y = \sqrt{(x+a)}, \quad \frac{dy}{dx} = \frac{1}{2\sqrt{(x+a)}}.$$

$$(iv.) \quad y = \frac{1}{\sqrt{(a-x)}}, \quad \frac{dy}{dx} = \frac{1}{2(a-x)^{\frac{3}{2}}}.$$

$$(v.) \quad y = \sqrt{(a^2+x^2)}, \quad \frac{dy}{dx} = \frac{x}{\sqrt{(a^2+x^2)}}.$$

$$(vi.) \quad y = \frac{1}{\sqrt{(a^2-x^2)}}, \quad \frac{dy}{dx} = \frac{x}{(a^2-x^2)^{\frac{3}{2}}}.$$

$$(vii.) \quad y = \frac{x}{\sqrt{(x^2+a^2)}}, \quad \frac{dy}{dx} = \frac{a^2}{(x^2+a^2)^{\frac{3}{2}}}.$$

(Solution.—Employing the rule of § 12,

$$\frac{dy}{dx} = \frac{\sqrt{(x^2+a^2)} - x \frac{x}{\sqrt{(x^2+a^2)}}}{x^2+a^2} = \frac{a^2}{(x^2+a^2)^{\frac{3}{2}}}).$$

$$(viii.) \quad y = \sqrt{(2ax-x^2)}, \quad \frac{dy}{dx} = \frac{a-x}{\sqrt{(2ax-x^2)}}.$$

(2) Differentiate with respect to x ,

$$\frac{1}{7}x^7 - \frac{2}{5}a^2x^5 + a^4x^3 - a^6x, \quad x\sqrt{(1-x^2)}, \quad x(1-2x^2)\sqrt{(1-x^2)},$$

$$\frac{a-x}{a+x}, \quad \sqrt{\frac{a-x}{a+x}}, \quad \frac{2x}{1-x^2}, \quad \frac{3x+x^3}{1+3x^2}, \quad \frac{4x-4x^3}{1-6x^2+x^4}, \quad \frac{5x+10x^3+x^5}{1+10x^2+5x^4}.$$

(3) Deduce the rule for the differentiation of a quotient from the rule for the differentiation of a product.

13. *Implicit and Explicit Relations between two variables, x and y .*

When the variables x and y are connected together by any relation, for instance by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the general equation of a conic section, or by

$$x^3 - 3axy + y^3 = 0,$$

the equation of a *cubic* curve, this relation is called an *implicit relation* between x and y , and is denoted generically by the notation

$$F(x, y) = 0 \dots \dots \dots (A).$$

When however it is possible by solution of the equation to obtain y in terms of x , or x in terms of y , then y is called an *explicit function* of x , or x of y , denoted generically as before by $y = fx$, or $x = \phi y$ (in the notation of *inverse functions*, § 25, $x = f^{-1}y$).

In the above implicit relations between x and y , we cannot obtain y explicitly in terms of x , or x in terms of y , except by the solution of a quadratic or cubic equation; but, for instance, from the implicit relation

$$x^2y^2 - a^2(x^2 - y^2) = 0,$$

the equation of a *quartic* curve, we obtain *explicitly*, by mere transposition,

$$y^2 = \frac{a^2x^2}{a^2 + x^2}, \quad \text{or} \quad x^2 = \frac{a^2y^2}{a^2 - y^2}.$$

From an implicit relation (A) between x and y to find dy/dx , it is not necessary to express y as an explicit function of x ; but we differentiate (A) with respect to x , treating y as a function of x , and obtain what is called the *first derived equation* and thence determine dy/dx as a function of x and y , without solving the equation (A);

thus from the implicit relation or equation of a conic section we obtain the first derived equation.

$$2ax + 2hy + 2hx \frac{dy}{dx} + 2by \frac{dy}{dx} + 2g + 2f \frac{dy}{dx} = 0,$$

so that
$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}.$$

For changing the coordinates x, y of a point on the conic section to $x + \Delta x, y + \Delta y$, the coordinates of an adjacent point on the curve, then

$$\begin{aligned} F(x + \Delta x, y + \Delta y) = \\ a(x + \Delta x)^2 + 2h(x + \Delta x)(y + \Delta y) + b(y + \Delta y)^2 \\ + 2g(x + \Delta x) + 2f(y + \Delta y) + c = 0; \end{aligned}$$

and therefore, by subtracting $F(x, y) = 0$,

$$\begin{aligned} a(2x\Delta x + \Delta x^2) + 2h(\Delta x \cdot y + x\Delta y + \Delta x\Delta y) \\ + b(2y\Delta y + \Delta y^2) + 2g\Delta x + 2f\Delta y = 0; \end{aligned}$$

and dividing by Δx ,

$$\begin{aligned} a(2x + \Delta x) + 2h(y + x \frac{\Delta y}{\Delta x} + \Delta y) + b(2y + \Delta y) \frac{\Delta y}{\Delta x} \\ + 2g + 2f \frac{\Delta y}{\Delta x} = 0; \end{aligned}$$

which in the limit, when Δx and Δy are indefinitely small, becomes the first derived equation.

Again, from the implicit relation

$$x^3 - 3axy + y^3 = 0,$$

we obtain the first derived equation

$$3x^2 - 3ay - 3ax \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0,$$

so that

$$\frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2}.$$

Examples.—Find dy/dx from the implicit relations

- (1) $x^2 + y^2 = c^2$, (4) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
 (2) $(x-a)^2 + (y-b)^2 = c^2$. (5) $x^m y^n = a^{m+n}$.
 (3) $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$. (6) $x^5 + y^5 - 5a^2 x^2 y = 0$.

(7) Given the implicit relation

$$x^3 - 3axy + y^3 = 0,$$

prove that $dy/dx = 0$, when $x = a\sqrt[3]{2}$, $y = a\sqrt[3]{4}$;

$$dx/dy = 0, \text{ when } x = a\sqrt[3]{4}, y = a\sqrt[3]{2}.$$

(8) Find where $dy/dx = 0$ and where $dx/dy = 0$, in the curve $x^5 + y^5 - 5a^2 x^2 y = 0$.

14. *Tracing Curves.*

The student should at this stage be exercised in tracing or plotting simple curves from their equations, thus exhibiting to the eye the flow or march of the function $y = fx$, or the *graph* of the function fx (§ 5).

Most of these curves should be already familiar to the student who has studied the elements of *Analytical Geometry*, as given in Smith's *Conic Sections*.

The curves should be drawn to scale as carefully as possible; for this purpose the paper which can be bought ruled into small squares is useful.

By drawing these curves and interpreting geometrically the differential coefficients obtained from their equations, the student will begin to understand the use of the Calculus, always difficult to explain to the beginner.

Examples.—Draw the curves:

- (1) $y = 1, x, x^2, x^3, x^4, \sqrt{x}, \frac{1}{x}, \frac{1}{x^2}, \dots$ (on one diagram).
 (2) $y = x - x^2, x - x^3, x + x^3, x + \frac{1}{x}, \dots$

- (3) $y^2=1$, x , x^2 , x^3 , x^4 , \sqrt{x} , $\frac{1}{x}$, $\frac{1}{x^2}$, ... (on one diagram).
- (4) $y^2=1-x^2$, x^2-1 , x^2+1 , $x-x^2$, $x+x^2$, x^2-x , ...
- (5) $x+y=1$, $x^2+y^2=1$, $x^3+y^3=1$, $x^4+y^4=1$, $\sqrt{x}+\sqrt{y}=1$,
 $\frac{1}{x}+\frac{1}{y}=1$, $\frac{1}{x^2}+\frac{1}{y^2}=1$, ...
- (6) $x^2-y^2=1$, $y^2-x^2=1$, $\frac{1}{x^2}-\frac{1}{y^2}=1$, $\frac{1}{y^2}-\frac{1}{x^2}=1$.
- (7) $y=\frac{x-1}{x-2}$, $\frac{(x-1)(x-2)}{x-3}$, $\frac{(x-1)(x-2)}{(x-3)(x-4)}$, $\frac{(x-1)(x-3)}{(x-2)(x-4)}$, ...
- (8) $x^2-4x+2y=0$, $xy-2x-y=0$, $(y-x)^2=1-x^2$.
- (9) Prove the properties of the following curves and sketch the curves:—

- (i.) In $y^2=px$, a parabola, the subnormal MG is constant.
- (ii.) In $x^2+y^2=a^2$, a circle, the normal PG is constant.
- (iii.) In $x^2+y^2=2ay$, a circle, $OT=TP$.
- (iv.) In $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, the part of the tangent TV intercepted by the axes is of constant length.
- (v.) In $by^2=x^3$, a semicubical parabola,
 $TM^2=\frac{8}{27}b \cdot MG$; and generally in $y^m=cx^n$,
 MG varies as $(TM)^{(2m-n)/n}$.
- (vi.) In $\frac{1}{x^n}+\frac{1}{y^n}=\frac{1}{a^n}$, $OT=\frac{x^{n+1}}{a^n}$; and determine p , the perpendicular from O upon the tangent.

(10) Prove that the equation of the tangent at (xy) of

- (i.) The circle (fig. 13) $x^2+y^2=c^2$ is $xx'+yy'=c^2$;
 or of the circle $(x-a)^2+(y-b)^2=c^2$ is

$$(x-a)(x'-a)+(y-b)(y'-b)=c^2;$$

- (ii.) The ellipse (fig. 8) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$;

(Solution :—Forming the first derived equation,

$$\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0, \text{ so that } \frac{dy}{dx} = -\frac{b^2x}{a^2y};$$

and then the equation of the tangent TPV is

$$y' - y = -\frac{b^2x}{a^2y}(x' - x), \quad \text{or} \quad \frac{xx'}{a^2} + \frac{yy'}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.)$$

(iii.) The *hyperbola* (fig. 14) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1.$

(iv.) The *parabola* (fig. 2 ii.) $y^2 = px$ is $\frac{y'}{\frac{1}{2}y} - \frac{x'}{x} = 1.$

(v.) The *hyperbola* (fig. 2 iii.) $xy = c^2$ is $\frac{x'}{2x} + \frac{y'}{2y} = 1.$

(This is called the *isothermal curve*, as it exhibits the relation between x the volume and y the pressure of a given quantity of gas at a constant temperature.)

(vi.) $x^m y^n = a^{m+n}$ is $\frac{x'}{(1+n/m)x} + \frac{y'}{(1+m/n)y} = 1.$

(This is called the *adiabatic* or *steam curve*, when we put for instance $m=7$ and $n=5$, or $m=10$ and $n=9$.)

(vii.) $\left(\frac{x}{a}\right)^m = \left(\frac{y}{b}\right)^n$ is $\frac{x'}{(1-n/m)x} + \frac{y'}{(1-m/n)y} = 1.$

Determine also the equation of the normal of these curves.

(11) Prove that the curves $x^m y^n = a^{m+n}$ and $\frac{x^2}{m} - \frac{y^2}{n} = b^2$ cut at right angles.

(12) Prove that the equation of the tangent to the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $\frac{x'}{\sqrt{x}} + \frac{y'}{\sqrt{y}} = \sqrt{a}$, and then $OT + OV = a$; and generally, the equation of the tangent of $x^n + y^n = a^n$ is $x^{n-1}x' + y^{n-1}y' = a^n$.

(12) Prove that if

$$(i.) \quad 9ay^2 = 4x^3, \quad \frac{ds}{dx} = \sqrt{\left(1 + \frac{x}{a}\right)};$$

$$(ii.) \quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}, \quad \frac{ds}{dx} = \left(\frac{a}{x}\right)^{\frac{1}{3}};$$

$$(iii.) \quad y^2 = 2lx, \quad \frac{ds}{dy} = \sqrt{\left(1 + \frac{y^2}{l^2}\right)}.$$

(13) Prove that the conic section

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

can be described with component velocities

$$dx/dt = hx + by + f, \quad dy/dt = -ax - hy - g,$$

$$\text{or } dx/dt = \sqrt{\{(h^2 - ab)x^2 + 2(fh - bg)x + f^2 - bc\}},$$

$$dy/dt = -\sqrt{\{(h^2 - ab)y^2 + 2(gh - af)y + g^2 - ac\}}.$$

Prove also that the equation of the tangent at (xy) is

$$(ax + hy + g)x' + (hx + by + f)y' + gx + fy + c = 0.$$

15. *Algebraical and Transcendental Functions.*

So far we have dealt only with *algebraical* functions of x , an algebraical function being defined as one which is *composed of a finite number of the algebraical operations of Addition, Subtraction, Multiplication, Division, Involution, and Evolution.*

An algebraical function of x may be defined in the most general manner as *the root of an algebraical equation of integral degree, the coefficients of which are rational integral algebraical functions of x* ; and a rational integral algebraical function of x is defined as *the sum of a finite number of terms, multiples of integral powers of x , such as*

$$ax^n + bx^{n-1} + cx^{n-2} + \dots,$$

where n is a positive integer; and then n , the highest power of x , is called the *degree* of the function.

But now we pass on to the functions of Trigonometry, such as $\sin x$, $\cos x$, ... , which are called *transcendental functions*; every function of x which is not an *algebraical* function being called a *transcendental* function.

Suppose we require to differentiate $\sin x$; then from the definition of a differential coefficient in § 3,

$$\begin{aligned}\frac{d \sin x}{dx} &= \text{lt} \frac{\sin(x+h) - \sin x}{h} = \text{lt} \frac{2 \cos(x + \frac{1}{2}h) \sin \frac{1}{2}h}{h} \\ &= \text{lt} \cos(x + \frac{1}{2}h) \frac{\sin \frac{1}{2}h}{\frac{1}{2}h} = \cos x;\end{aligned}$$

since $\text{lt} \cos(x + \frac{1}{2}h) = \cos x$, and $\text{lt}(\sin \frac{1}{2}h)/\frac{1}{2}h = 1$, when x and therefore h is expressed in circular measure.

Or, more simply from the definition of § 5,

$$\frac{d \sin x}{dx} = \text{lt} \frac{\sin(x+h) - \sin(x-h)}{2h} = \text{lt} \cos x \frac{\sin h}{h} = \cos x;$$

and so on for the other functions of Trigonometry.

But if x is given in degrees, or minutes, or seconds, then

$$\begin{aligned}\frac{d \sin x^\circ}{dx} &= \frac{\pi}{180} \cos x^\circ, \quad \frac{d \sin x'}{dx} = \frac{\pi}{180 \times 60} \cos x', \\ \frac{d \sin x''}{dx} &= \frac{\pi}{180 \times 60 \times 60} \cos x'';\end{aligned}$$

so that extraneous factors are avoided by the use of circular measure.

We thus require a knowledge of the circular measure of an angle and of the formulas of Trigonometry, a subject with which the student is presumably already familiar, but on which it will be desirable to make a slight digression, so far as is required for our purposes.

16. *Trigonometry. The Circular Measure, and the Circular Functions of an Angle.*

Definitions.—The circular measure (θ) of an angle AOP is the number giving the ratio of the circular arc AP to the radius OA .

The unit angle is the angle subtended by an arc equal to its radius; this angle is called the *radian*, and contains

$$180 \times 60 \times 60 \div \pi = 206264''\cdot 8, \text{ or } 57^\circ 17' 44''\cdot 8.$$

Here π denotes as usual the ratio of the circumference to the diameter, so that $2\pi a$ is the circumference of a circle of radius a ; thus the c.m. (circular measure) of a right angle is $\frac{1}{2}\pi$ (radians), of the angle of an equilateral triangle is $\frac{1}{3}\pi$, and so on; and $\pi = 3\cdot 14159265\dots$, an *incommensurable* number.

By Euclid VI. 33, the area of the sector OAP is to that of the whole circle as the arc AP is to the circumference, or as θ to 2π ; and the area of the circle being πa^2 , the area of the sector is $\frac{1}{2}a^2\theta$, or one half the rectangle of the radius and the arc.

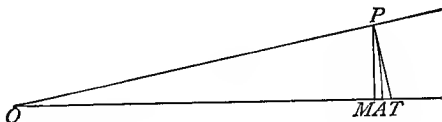


Fig. 5

In fig. 5 draw PM the perpendicular from P on OA , and let the tangent at P meet OA produced in T ; then according to the definitions of Trigonometry,

$$\frac{\text{arc } AP}{OP} = \theta, \quad \frac{OM}{OP} = \cos \theta, \quad \frac{MP}{OP} = \sin \theta,$$

$$\frac{PT}{OP} = \tan \theta, \quad \frac{OT}{OP} = \sec \theta, \quad \frac{AM}{AO} = \text{vers } \theta;$$

also if TP produced meets OB drawn perpendicular to OA in V ,

$$\frac{PV}{OP} = \cot \theta, \quad \frac{OV}{OP} = \text{cosec } \theta;$$

we shall call these functions of θ the *circular functions*.

Then the triangle $OAP = \frac{1}{2}a^2 \sin \theta$, the sector $OAP = \frac{1}{2}a^2\theta$, and the triangle $OPT = \frac{1}{2}a^2 \tan \theta$; and these are obviously in ascending order of magnitude, so that

$$\sin \theta, \theta, \tan \theta,$$

are also in ascending order, when θ is the c.m. of an angle less than a right angle, that is when $\theta < \frac{1}{2}\pi$.

Now make θ indefinitely small; then since, when $\theta = 0$,

$$\text{lt } \sin \theta / \tan \theta = \text{lt } \cos \theta = 1;$$

therefore also, $\text{lt}(\sin \theta)/\theta = 1$, and $\text{lt}(\tan \theta)/\theta = 1$;

when $\theta = 0$; as assumed in the last article.

More generally, when $\theta = 0$,

$$\frac{\sin m\theta}{n\theta} = \frac{\sin m\theta}{m\theta} \frac{m}{n} = \frac{m}{n}, \quad \frac{\tan m\theta}{n\theta} = \frac{m}{n};$$

thus, when $h = 0$, $\frac{\sin h^\circ}{h^\circ} = \frac{\tan h^\circ}{h^\circ} = \frac{\pi}{180}$,

$$\frac{\sin h'}{h'} = \frac{\tan h'}{h'} = \frac{\pi}{180 \times 60}, \quad \frac{\sin h''}{h''} = \frac{\tan h''}{h''} = \frac{\pi}{180 \times 60 \times 60}.$$

Again, when $\theta = 0$,

$$\text{lt } \frac{\text{vers } \theta}{\theta} = \text{lt } \frac{AM}{\text{arc } AP} = \text{lt } \frac{AM}{AP} \text{lt } \frac{\text{chord } AP}{\text{arc } AP} = 0,$$

because $\text{lt } \frac{\text{chord } AP}{\text{arc } AP} = 1$, and $\text{lt } \frac{AM}{AP} = \text{lt } \sin \frac{1}{2}\theta = 0$.

As a numerical verification, the student may calculate from the Tables the values of $(\sin \theta)/\theta$, $(\tan \theta)/\theta$, and $(\text{vers } \theta)/\theta$, when θ is the c.m. of an angle of 3° , 2° , 1° , $30'$, $10'$, $1'$, $30''$, $10''$, $1''$.

When $\theta = \pi/n$, the c.m. of $180/n$ degrees, then $2MP$ and $2PT$ are sides of regular polygons of n sides, inscribed and circumscribed to the circle; so that $2na \sin \theta/n$ and $2na \tan \theta/n$ are the perimeters of the inscribed and circumscribed polygons of n sides; and the perimeter of the circle $2\pi a$ being intermediate in length, we can, by

sufficiently increasing n , determine closer and closer limits between which π lies; in this manner Archimedes found that π lies between $3\frac{1}{7}$ and $3\frac{1}{4}$.

As a numerical exercise the student may calculate limits between which π lies, by taking $n=180$, 180×60 , ... in $n \sin \theta/n$ and $n \tan \theta/n$.

17. Differentiation of the Circular Functions.

Having thus justified the differentiation of $\sin x$,

$$\frac{d \sin x}{dx} = \cos x,$$

where x is the circular measure of a variable angle, we proceed similarly with the other functions:—

$$\begin{aligned} \frac{d \cos x}{dx} &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \rightarrow 0} \frac{-2 \sin(x + \frac{1}{2}h) \sin \frac{1}{2}h}{h} \\ &= -\lim_{h \rightarrow 0} \sin(x + \frac{1}{2}h) \frac{\sin \frac{1}{2}h}{\frac{1}{2}h} = -\sin x. \end{aligned}$$

$$\begin{aligned} \frac{d \tan x}{dx} &= \lim_{h \rightarrow 0} \frac{\tan(x+h) - \tan x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin h}{h} \sec(x+h) \sec x = \sec^2 x. \end{aligned}$$

$$\begin{aligned} \frac{d \cot x}{dx} &= \lim_{h \rightarrow 0} \frac{\cot(x+h) - \cot x}{h} \\ &= -\lim_{h \rightarrow 0} \frac{\sin h}{h} \operatorname{cosec}(x+h) \operatorname{cosec} x = -\operatorname{cosec}^2 x. \end{aligned}$$

$$\begin{aligned} \frac{d \sec x}{dx} &= \lim_{h \rightarrow 0} \frac{\sec(x+h) - \sec x}{h} = \lim_{h \rightarrow 0} \frac{\cos x - \cos(x+h)}{h \cos(x+h) \cos x}, \\ &= \sin x / \cos^2 x = \sec x \tan x. \end{aligned}$$

Similarly

$$\frac{d \operatorname{cosec} x}{dx} = -\operatorname{cosec} x \cot x, \quad \frac{d \operatorname{vers} x}{dx} = \sin x.$$

The formulas of Trigonometry we have assumed in these differentiations are, with the usual notation,

Thus if x is the c.m. of AOR , and h of the small angles POR, QOR ; then employing the definition of § 5,

$$\begin{aligned}\frac{d \sin x}{dx} &= \text{lt} \frac{\sin(x+h) - \sin(x-h)}{2h} = \text{lt} \frac{LP - MQ}{\text{arc } PQ} \\ &= \text{lt} \frac{2pP}{\text{arc } PQ} = \text{lt} \frac{pP}{PR} \cdot \frac{\text{chord } PQ}{\text{arc } PQ} = \cos x. \\ \frac{d \cos x}{dx} &= \text{lt} \frac{OL - OM}{\text{arc } PQ} \\ &= -\text{lt} \frac{2pR}{\text{arc } PQ} = -\text{lt} \frac{pR}{PR} \cdot \frac{\text{chord } PQ}{\text{arc } PQ} = -\sin x.\end{aligned}$$

With the definition of § 3,

$$\begin{aligned}\frac{d \tan x}{dx} &= \text{lt} \frac{\tan(x+h) - \tan x}{h} = \text{lt} \frac{tu}{\text{arc } rP} \\ &= \text{lt} \frac{tu}{tv} \cdot \frac{tv}{\text{arc } tw} \cdot \frac{\text{arc } tw}{\text{arc } rP} = \text{lt} \frac{Ou}{OA} \cdot \frac{Ot}{OA} = \sec^2 x;\end{aligned}$$

since $\frac{tu}{tv} = \frac{Ou}{OA}$, $\text{lt} \frac{tv}{\text{arc } tw} = 1$ (§ 16), and $\frac{\text{arc } tw}{\text{arc } rP} = \frac{Ot}{OA}$.

$$\begin{aligned}\frac{d \sec x}{dx} &= \text{lt} \frac{wu}{\text{arc } rP} = \text{lt} \frac{wu}{vu} \cdot \frac{vu}{tv} \cdot \frac{tv}{\text{arc } tw} \cdot \frac{\text{arc } tw}{\text{arc } rP} \\ &= \tan x \sec x; \text{ since}\end{aligned}$$

$$\text{lt} \frac{wu}{vu} = 1, \text{lt} \frac{vu}{tv} = \text{lt} \frac{Au}{OA} = \tan x, \text{lt} \frac{tv}{\text{arc } tw} = 1, \frac{\text{arc } tw}{\text{arc } rP} = \sec x.$$

19. By means of the formula of § 11 for the differentiation of a function of a function, we can differentiate any power of the circular functions of x , and the circular functions of any function of x ; thus

$$\begin{aligned}d(\sin x)^m/dx &= m(\sin x)^{m-1} \cos x; \quad d \sin y/dx = \cos y dy/dx; \\ d(\sin y)^m/dx &= m(\sin y)^{m-1} \cos y dy/dx.\end{aligned}$$

As an exercise, the student may in the same manner differentiate with respect to x the functions $(\cos x)^n$, $(\tan x)^n$, $(\cot x)^n$, $(\sec x)^n$, $(\text{cosec } x)^n$, $(\text{vers } x)^n$, $\{f(x)\}^n$, $\sin fx$, $\{f(\sin x)\}^n$, $(\cos fx)^n, \dots$

Examples.—(1) Prove the following differentiations :

- (i.) $y = \frac{1}{2}x + \frac{1}{4}\sin 2x$, $dy/dx = \cos^2 x$,
 (ii.) $y = \frac{1}{2}x - \frac{1}{4}\sin 2x$, $dy/dx = \sin^2 x$.
 (iii.) $y = \frac{1}{3}\cos^3 x - \cos x$, $dy/dx = \sin^3 x$.
 (iv.) $y = \sin x - \frac{1}{3}\sin^3 x$, $dy/dx = \cos^3 x$.
 (v.) $y = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$, $dy/dx = \cos^4 x$,
 and write down y when $dy/dx = \sin^4 x$.
 (vi.) $y = \tan x - x$, $dy/dx = \tan^2 x$.
 (vii.) $y = -\cot x - x$, $dy/dx = \cot^2 x$.
 (viii.) $y = \frac{1}{3}\tan^3 x - \tan x + x$, $dy/dx = \tan^4 x$.
 (ix.) $y = \frac{1}{3}\tan^3 x + \tan x$, $dy/dx = \sec^4 x$.

(2) From the definition of §3, differentiate with respect to x ,
 $\sin 2x$, $\cos x/a$, $\tan(mx+n)$, $x \tan x$, $\tan x^2$, $\sin x^n$.

(3) Differentiate x^n , $\sin x$, $\tan x^n$, with respect to x^m .

20. It will be useful also to draw the graphs of the circular functions ; thus the equations

$$y/b = \pm \sin 2\pi x/a \text{ and } \pm \cos 2\pi x/a$$

represent in fig. 7, for two values of b , the plan of the spirals of a double-threaded screw of pitch a .

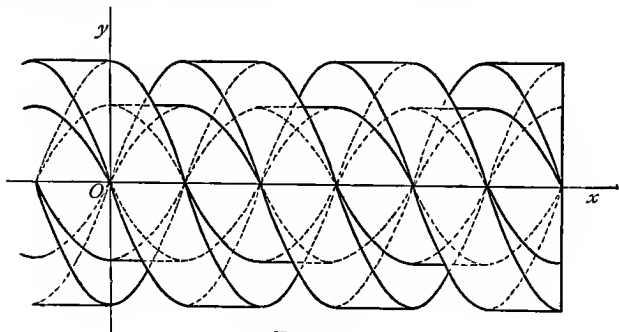


Fig. 7

Examples.—Draw the curves: (1) $y = \sin x, \cos x, \tan x, \cot x, \sec x, \operatorname{cosec} x, \operatorname{vers} x$ (on one diagram).

(2) $y = \sin^{-1}x, \cos^{-1}x, \tan^{-1}x, \cot^{-1}x, \sec^{-1}x, \operatorname{cosec}^{-1}x, \operatorname{vers}^{-1}x$ (on one diagram); and superpose them on the preceding diagram.

(3) $y = x \sin x, (\sin x)/x, (\sin x)^2, \sin x^2, \sin \sqrt{x}, x \cot \frac{1}{2}\pi x.$

(4) $\sin x = \sin y, \cos x = \cos y, \tan x = \tan y,$
 $(\sin x)^2 + (\sin y)^2 = 1, (\tan x)^2 + (\tan y)^2 = 1.$

(5) Prove that in the ellipse $(x/a)^2 + (y/b)^2 = 1$, we may put $x = a \cos \theta, y = b \sin \theta$ (θ is then called the *excentric angle*); and $AOp = \theta$, and the equations of the tangent and normal are (fig. 8)

$$(x/a)\cos \theta + (y/b)\sin \theta = 1; ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2.$$

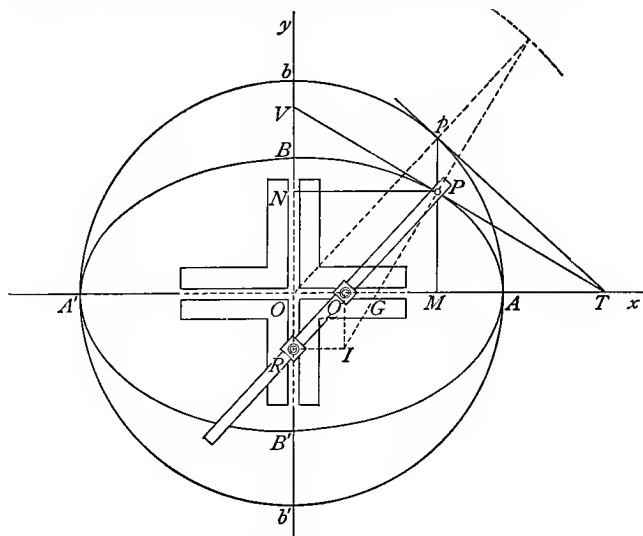


Fig.8

(Fig. 8 shows the *elliptic trammel* or *compasses*, the theory of which is obvious from this example.)

- (6) Prove that in the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, we may put $x = a(\cos \theta)^{\frac{2}{3}}$, $y = a(\sin \theta)^{\frac{2}{3}}$; and then $p = \sqrt[3]{axy}$; and the equations of the tangent and normal are $x \sec \theta + y \operatorname{cosec} \theta = a$; $x \cos \theta - y \sin \theta = a \cos 2\theta$.

21. The Cycloid.

As an illustration of the use of a variable angle or *parameter* θ for expressing the coordinates x and y of a point on a curve, consider the *cycloid*, the curve traced out by a point in the circumference of a circle which is rolling on a plane; a curve often seen described by a piece of paper sticking to the rim of a carriage wheel.

Starting from the origin O where the point is originally in contact with a horizontal plane, the point P describes the curve OP , and when the wheel has turned through an angle θ , the point P will have ascended a vertical height $MP = a \operatorname{vers} \theta$, while P will have advanced horizontally $OM = a\theta - a \sin \theta$, a denoting the radius of the wheel (fig. 9).

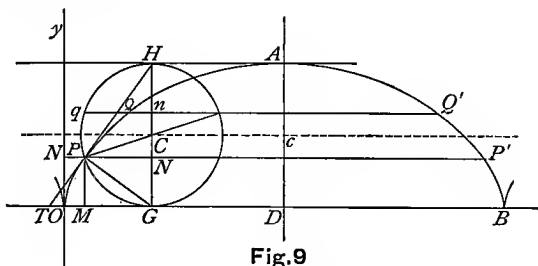


Fig. 9

Thus $x = OM = a(\theta - \sin \theta)$, $y = MP = a \operatorname{vers} \theta$; and the elimination of θ gives

$$x = a \operatorname{vers}^{-1} y / a - \sqrt{(2ay - y^2)},$$

the equation of the cycloid in x and y .

The complicated nature of this equation makes it preferable to retain the angle θ ; and now

$$\frac{dx}{d\theta} = a \text{ vers } \theta = y, \quad \frac{dy}{d\theta} = a \sin \theta,$$

$$\tan \psi = \frac{dy}{dx} = \frac{\sin \theta}{\text{vers } \theta} = \cot \frac{1}{2}\theta, \text{ or } \psi = \frac{1}{2}\pi - \frac{1}{2}\theta,$$

so that PG , the normal to the curve, passes through G , the point of contact of the wheel with the ground.

This is obvious if we consider that the wheel is instantaneously turning about G , that is, G is the *centre of instantaneous rotation*; and then GP is the normal, and TPH is the tangent.

$$\begin{aligned} \text{Again } \frac{ds}{d\theta} &= \sqrt{\left(\frac{dx^2}{d\theta^2} + \frac{dy^2}{d\theta^2}\right)} \\ &= a\sqrt{(\text{vers}^2\theta + \sin^2\theta)} = 2a \sin \frac{1}{2}\theta; \\ \frac{ds}{dy} &= \text{cosec } \psi = \sec \frac{1}{2}\theta = \left(\frac{2a}{2a-y}\right)^{\frac{1}{2}}. \end{aligned}$$

Any other point P fixed on the wheel at a distance b from the centre will describe a curve in which $x = a\theta - b \sin \theta$, $y = a - b \cos \theta$; these curves are called *trochoids*; and GP is still the normal at P .

For a point on one of the spokes of a wheel, $b < a$, and the trochoid is called *curtate*. When $b > a$, the trochoid is called *prolate*; a point on one of the floats of a paddle wheel describes a prolate trochoid.

22. Polar Coordinates.

If OP is denoted by r and the angle xOP by θ , where P is any variable point on a curve AP (fig. 10), then r, θ are called the *polar coordinates* of P .

They are connected with the former coordinates (x, y) of § 5 by the relations $x = r \cos \theta$, $y = r \sin \theta$; or $r = \sqrt{(x^2 + y^2)}$, $\theta = \tan^{-1}y/x$.

If $r=f\theta$ is the polar equation of the curve AP , and if $xOP=\theta$, then $OP=r=f\theta$; so that the curve AP is, with polar co-ordinates, the graph of the function $f\theta$.

Such graphs are required, for instance, in plotting out the turning moment of an engine at any part of the revolution, or in drawing a cam in mechanism.

When P moves to an adjacent point p on the curve, suppose the polar co-ordinates r, θ to receive increments $\Delta r, \Delta\theta$; so that $r+\Delta r, \theta+\Delta\theta$ are the polar co-ordinates of p ; then $xOp=\theta+\Delta\theta$, $POp=\Delta\theta$; and

$$Op=f(\theta+\Delta\theta)=r+\Delta r;$$

$$Rp=f(\theta+\Delta\theta)-f\theta=\Delta r;$$

also

$$PR=r\Delta\theta;$$

if PR is the arc of a circle described with centre O .

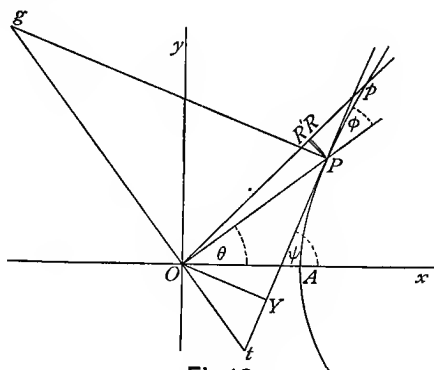


Fig.10

Draw the straight line PR' perpendicular to Op ; then $\lim \frac{PR'}{PR} = \lim \frac{\sin \Delta\theta}{\Delta\theta} = 1$, by § 16; and $\lim \frac{R'R}{PR} = \lim \frac{\text{vers } \Delta\theta}{\Delta\theta} = 0$, so that $\lim \frac{R'p}{Rp} = 1$.

(In the language of *Infinitesimals*, PR and PR' are called infinitesimals of the *first order*, and $R'R$ an infinitesimal of the *second order*.)

Now if the angle between OP produced and the curve Pp , or the tangent at P , is denoted by ϕ , then ϕ is called the *radial angle*, and

$$\begin{aligned}\tan \phi &= \text{lt} \tan OpP = \text{lt} \frac{PR'}{R'p} = \text{lt} \frac{PR'}{PR} \cdot \frac{Rp}{R'p} \cdot \frac{PR}{Rp} = \text{lt} \frac{PR}{Rp} \\ (\text{since } \text{lt} \frac{PR'}{PR} &= 1, \text{ and } \text{lt} \frac{R'p}{Rp} = 1) \\ &= \text{lt} \frac{r\Delta\theta}{\Delta r} = \frac{rd\theta}{dr};\end{aligned}$$

$$\text{and} \quad \cot \phi = \frac{dr}{rd\theta} = \frac{d \log r}{d\theta},$$

anticipating the result of § 28.

23. If the arc AP , measured from any fixed point A up to the variable point P , is denoted by s , and if Pp , the increment of the arc, is denoted by Δs ; then since, as in § 9,

$$\text{lt} \frac{\text{chord } Pp}{\text{arc } Pp} = 1,$$

$$\begin{aligned}\text{therefore } \cos \phi &= \text{lt} \cos OpP = \text{lt} \frac{R'p}{Pp} = \text{lt} \frac{R'p}{Rp} \cdot \frac{\text{arc } Pp}{\text{chord } Pp} \cdot \frac{Rp}{\text{arc } Pp} \\ &= \text{lt} \frac{Rp}{\text{arc } Pp} \quad (\text{since } \text{lt} \frac{R'p}{Rp} = 1, \text{ and } \text{lt} \frac{\text{chord } Pp}{\text{arc } Pp} = 1) \\ &= \text{lt} \frac{\Delta r}{\Delta s} = \frac{dr}{ds}.\end{aligned}$$

$$\text{Similarly} \quad \sin \phi = \text{lt} \sin OpP = \text{lt} \frac{PR'}{Pp}$$

$$= \text{lt} \frac{PR}{\text{arc } Pp} = \text{lt} \frac{r\Delta\theta}{\Delta s} = \frac{rd\theta}{ds},$$

$$\text{Therefore} \quad \frac{dr^2}{ds^2} + \frac{r^2 d\theta^2}{ds^2} = \cos^2 \phi + \sin^2 \phi = 1.$$

If Ot is drawn at right angles to OP to meet the tangent at P in t , then Ot is called the *polar subtangent*; and

$$Ot = r \tan \phi = r^2 d\theta/dr.$$

It is convenient for subsequent purposes to represent the reciprocal of r by u ; and then $r = \frac{1}{u}$, $\frac{dr}{d\theta} = -\frac{1}{u^2} \frac{du}{d\theta}$; so that $Ot = -d\theta/du$.

If tO produced meets the normal at P in g , then Og is called the *polar subnormal*; and

$$Og = r \cot \phi = dr/d\theta.$$

Looking along OP from O , t will be to the right and g to the left when $dr/d\theta$ is positive; and *vice versa* if $dr/d\theta$ is negative.

The tangent $Pt = r \sec \phi = r ds/dr$; and the normal $Pg = r \operatorname{cosec} \phi = ds/d\theta$; and denoting the perpendicular OY from the origin upon the tangent at P by p , then $p = r \sin \phi = r^2 d\theta/ds$.

Again, from fig. 10,

$$\frac{1}{OY^2} = \frac{1}{Ot^2} + \frac{1}{OP^2},$$

therefore

$$\frac{1}{p^2} = \left(\frac{du}{d\theta}\right)^2 + u^2,$$

a useful expression for p .

Employing a dynamical interpretation as before in § 10, and supposing P to move in the curve AP (like a planet round the sun at O) so that its polar co-ordinates r , θ are given functions of the time t ; then the component velocities of P in the direction OP and in the direction PR perpendicular to OP are, by resolution,

$$\frac{ds}{dt} \cos \phi = \frac{dr}{dt} \text{ and } \frac{ds}{dt} \sin \phi = \frac{r d\theta}{dt}, \text{ respectively;}$$

these are called the *radial* and *transversal* velocities of

P , the resultant velocity being ds/dt in the direction of the tangent tP .

Therefore
$$\frac{ds^2}{dt^2} = \frac{dr^2}{dt^2} + \frac{r^2 d\theta^2}{dt^2};$$

and by making $t = \theta$, or r , we obtain the relations

$$\frac{ds^2}{d\theta^2} = \frac{dr^2}{d\theta^2} + r^2,$$

$$\frac{ds^2}{dr^2} = 1 + \frac{r^2 d\theta^2}{dr^2};$$

while making $t = s$ gives, as before,

$$1 = \frac{dr^2}{ds^2} + \frac{r^2 d\theta^2}{ds^2}.$$

It is important to have the power of plotting curves from the equations in polar co-ordinates, so the following examples should be worked, as preliminary practice.

Examples.—Draw the following curves whose equations are given in polar co-ordinates:—

- (1) $r = 1, \theta, \theta^2, \sqrt{\theta}, \theta^{-\frac{1}{2}}, \theta^{-1}, \theta^{-2} \dots$
- (2) $r = \cos \theta, \cos 2\theta, \cos 3\theta, \cos 4\theta.$
- (3) $r = \sin \theta, \sin 2\theta, \sin 3\theta, \sin 4\theta.$
- (4) $r = \sec \theta, \operatorname{cosec} \theta, \tan \theta, \cot \theta.$
- (5) $r = \cos \frac{1}{2}\theta, \sin \frac{1}{2}\theta, \cos \frac{1}{3}\theta, \cos \frac{2}{3}\theta, \cos \frac{1}{4}\theta, \cos \frac{3}{4}\theta, \cos \frac{5}{4}\theta, \cos \frac{3}{2}\theta, \cos \frac{5}{3}\theta, \sec \frac{1}{2}\theta, \operatorname{cosec} \frac{1}{2}\theta.$
- (6) $r^2 = \sin 2\theta, \cos 2\theta, \sec 2\theta, \operatorname{cosec} 2\theta.$
- (7) $r = \operatorname{vers} \theta, 1 + \cos \theta, (\operatorname{vers} \theta)^{-1}, (1 + \cos \theta)^{-1}, \sec^2 \frac{1}{2}\theta, \operatorname{cosec}^2 \frac{1}{2}\theta, (1 - \cos a \cos \theta)^{-1}, (\cos a - \cos \theta)^{-1}, \cos \theta - \cos a, 1 - \cos a \cos \theta$ (take $a = \frac{1}{3}\pi$).
- (8) $r = a \cos \theta + b \sin \theta, a \sec \theta + b \operatorname{cosec} \theta;$

$$\frac{1}{r} = \frac{\cos \theta}{a} + \frac{\sin \theta}{b}, \frac{1}{r^2} = \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}.$$

(9) Prove that in the curves

(i.) $r = a$, $\phi = \frac{1}{2}\pi$.

(ii.) $r = a \sin \theta$, $\phi = \theta$.

(iii.) $r = b \cos \theta$, $\phi = \frac{1}{2}\pi + \theta$.

(iv.) $r^n = a^n \sin n\theta$, $\phi = n\theta$, and $p = r^{n+1}/a^n$.

(v.) $r^n = b^n \cos n\theta$, $\phi = \frac{1}{2}\pi + n\theta$;

and the curves (iv.) and (v.) cut at right angles.

(vi.) $r^n = a^n \sec n\theta$, $\phi = \frac{1}{2}\pi - n\theta$, and $p = a^n/r^{n-1}$.

(vii.) $r^n = b^n \operatorname{cosec} n\theta$, $\phi = \pi - n\theta$;

and the curves (vi.) and (vii.) cut at right angles.

(10) Prove that the curves

$$r^n = a^n \cos(n\theta - \alpha) \text{ and } r^n = a^n \cos(n\theta - \beta)$$

cut at an angle $\alpha - \beta$.

(11) Prove that

Ot is constant in $r = a/\theta$ (the *reciprocal spiral*)
and $Og = a\theta$;

Og is constant in $r = a\theta$ (the *spiral of Archimedes*, fig. 12, ii.) and $Ot = a\theta^2$;

Pg is constant in $r = a \sin \theta$ (the circle).

Find the locus of g and of t in the circle $r = a \cos \theta$.

(12) Prove that the locus of t is the straight line $lu = e \cos \theta$
in the curve $lu = l/r = 1 + e \cos \theta$ (a *conic section*).

Prove also that if GL is drawn perpendicular to OP
(fig. 10), then $PL = l$.

(13) Prove that if ϕ_1 and ϕ_2 are the radial angles of the
locus of t and of g , $\cot \phi_1 + \cot \phi_2 = 2 \cot \phi$.

(14) Prove that if, with given elevation of a gun to
the horizon, the range of the projectile on the
level is r yards, and if on a slight descending slope
of one in m the range is increased by Δr yards,
the cotangent of the angle of descent is $m\Delta r/r$,
approximately.

(15) Given that $5'$ extra elevation or depression of a gun increases or diminishes the range of r yards by Δr , prove that the cotangent of the angle of descent is about $\Delta r \cot 5'/r$, and the slope of the descent is one in 688 $\Delta r/r$.

(16) Prove that in

(i.) $r = a \operatorname{vers} \theta$ (the *cardioid*), $ds/d\theta = 2a \sin \frac{1}{2}\theta$.

(ii.) $r = 2a/(\operatorname{vers} \theta)$ (the *parabola*), $ds/d\theta = a(\operatorname{cosec} \frac{1}{2}\theta)^3$.

(iii.) $r = a\theta$, $\frac{ds}{dr} = \sqrt{1 + \frac{r^2}{a^2}}$. (iv.) $r = a/\theta$, $\frac{ds}{dr} = \sqrt{1 + \frac{a^2}{r^2}}$.

24. It has already been assumed in § 5 that dy/dx and dx/dy are reciprocal, or that their product is unity, y being any function of x and x therefore a function of y .

The proof, if any proof is required, may be given thus: if Δx is any increment of x and Δy the corresponding increment of y , then always

$$\frac{\Delta y}{\Delta x} \times \frac{\Delta x}{\Delta y} = 1;$$

and therefore proceeding to the limit,

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1,$$

or dy/dx and dx/dy are reciprocal; provided however the values of x and y are the same in each, in the case of *many valued* functions; that is, functions y of x , which for any given value of x have more than one value of y , or *vice versa*; thus, if $y = \sin^{-1}x$, or $x = \sin y$, then for any given value of x , y has any value comprised in the formula

$$n\pi + (-1)^ny.$$

This theorem is required in the

25. *Differentiation of the Inverse Circular Functions.*

If y is some function of x , denoted by fx , then x is some function of y , which it is convenient to denote by $f^{-1}y$; so that f and f^{-1} denote functions *inverse* to each other, such that $f^{-1}(fx) = x$, and $f(f^{-1}y) = y$.

Thus in Algebra x^n and $\sqrt[n]{x}$ are functions of x inverse to each other; for $(\sqrt[n]{x})^n = x$ and $\sqrt[n]{(y^n)} = y$.

In Trigonometry it is usual to employ the abbreviation $\sin^2 x, \sin^3 x, \dots$ for $(\sin x)^2, (\sin x)^3, \dots$ with positive powers; but $\sin^{-1} x$ is never used to mean $1/\sin x$ or $\operatorname{cosec} x$, but to denote the c.m. of an angle whose sine is the number x ; and so on for the other circular functions.

To differentiate $\sin^{-1} x$, let $\sin^{-1} x = y$,

then $x = \sin y$,

and $dx/dy = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$;

therefore $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$, or $\frac{d \sin^{-1} x}{dx} = \frac{1}{\sqrt{1 - x^2}}$.

Similarly $\frac{d \sin^{-1} x/a}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$.

Let $\cos^{-1} x = y, x = \cos y$,

$dx/dy = -\sin y = -\sqrt{1 - \cos^2 y} = -\sqrt{1 - x^2}$,

$\frac{dy}{dx} = \frac{d \cos^{-1} x}{dx} = -\frac{1}{\sqrt{1 - x^2}}$;

and $\frac{d \cos^{-1} x/a}{dx} = -\frac{1}{\sqrt{a^2 - x^2}}$.

Since $\cos^{-1} x/a + \sin^{-1} x/a = \frac{1}{2}\pi$,

therefore $\frac{d \cos^{-1} x/a}{dx} + \frac{d \sin^{-1} x/a}{dx} = 0$,

and $\frac{d \cos^{-1} x/a}{dx} = -\frac{d \sin^{-1} x/a}{dx} = -\frac{1}{\sqrt{a^2 - x^2}}$

thus illustrating the reason for this relation.

Again, let $\tan^{-1}x = y$,
 then $x = \tan y$,
 $dx/dy = \sec^2 y = 1 + \tan^2 y = 1 + x^2$,
 and $\frac{dy}{dx} = \frac{d \tan^{-1}x}{dx} = \frac{1}{1+x^2}$;
 also $\frac{d \tan^{-1}x/a}{dx} = \frac{a}{a^2+x^2}$.

Let $\cot^{-1}x = y$, $x = \cot y$,
 $dx/dy = -\operatorname{cosec}^2 y = -\cot^2 y - 1 = -x^2 - 1$,
 $\frac{d \cot^{-1}x}{dx} = -\frac{1}{x^2+1}$,
 and $\frac{d \cot^{-1}x/a}{dx} = -\frac{a}{x^2+a^2}$.
 Since $\cot^{-1}x/a + \tan^{-1}x/a = \frac{1}{2}\pi$,
 therefore $\frac{d \cot^{-1}x/a}{dx} + \frac{d \tan^{-1}x/a}{dx} = 0$,
 and $\frac{d \cot^{-1}x/a}{dx} = -\frac{d \tan^{-1}x/a}{dx} = -\frac{a}{a^2+x^2}$.

Let $\sec^{-1}x = y$, $x = \sec y$,
 $dx/dy = \sec y \tan y = x\sqrt{x^2-1}$,
 $\frac{d \sec^{-1}x}{dx} = \frac{1}{x\sqrt{x^2-1}}$.
 Similarly $\frac{d \operatorname{cosec}^{-1}x}{dx} = -\frac{1}{x\sqrt{x^2-1}}$.

Let $\operatorname{vers}^{-1}x/a = y$, $x = a \operatorname{vers} y$,
 $dx/dy = a \sin y = a\sqrt{1-\cos^2 y}$
 $= a\sqrt{2 \operatorname{vers} y - \operatorname{vers}^2 y} = \sqrt{(2ax - x^2)}$;
 $\frac{d \operatorname{vers}^{-1}x/a}{dx} = \frac{1}{\sqrt{(2ax - x^2)}}.$

By the rule of § 11 for the differentiation of a function of a function, if x is a function of t ,

$$\frac{d \sin^{-1} x}{dt} = \frac{d \sin^{-1} x}{dx} \frac{dx}{dt} = \frac{1}{\sqrt{1-x^2}} \frac{dx}{dt};$$

or $d \sin^{-1} x = dx / \sqrt{1-x^2}$,

in the notation of *differentials*; and so on for the remaining inverse functions.

26. The inverse circular functions are not much required in Elementary Trigonometry, but are indispensable in the Differential and Integral Calculus: so the principal formulas of Trigonometry, expressed by the direct functions, are given here on the opposite page, with the corresponding inverse notation.

Examples.—(1) Construct a Table giving any direct circular function in terms of any other, and differentiate them as a verification.

(2) Construct a similar Table for the inverse circular functions, and differentiate them.

(3) Given $y = \frac{1}{2} \cos^{-1}(2x^2 - 1)$, $\frac{1}{3} \cos^{-1}(4x^3 - 3x)$,
 $\frac{1}{4} \cos^{-1}(8x^4 - 8x^2 + 1), \dots$

prove that $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}.$

(4) Given $y = \frac{1}{2} \sin^{-1} 2x \sqrt{1-x^2}$, $\frac{1}{3} \sin^{-1}(3x - 4x^3)$,
 $\frac{1}{4} \sin^{-1} 4x(1-2x^2) \sqrt{1-x^2}, \dots$

prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}},$

(5) Given $y = \frac{1}{2} \tan^{-1} \frac{2x}{1-x^2}$, $\frac{1}{2} \sin^{-1} \frac{2x}{1+x^2}$, $\frac{1}{2} \cos^{-1} \frac{1-x^2}{1+x^2}$,

$\frac{1}{3} \tan^{-1} \frac{3x-x^3}{1-3x^2}$, $\frac{1}{4} \tan^{-1} \frac{4x-4x^3}{1-6x^2+x^4}$, $\frac{1}{5} \tan^{-1} \frac{5x-10x^3+x^5}{1-10x^2+5x^4}, \dots$

prove that $\frac{dy}{dx} = \frac{1}{1+x^2}.$

DIRECT FUNCTIONS.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\cos 2A = 2 \cos^2 A - 1$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\sin 2A = 2 \sin A \cos A$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\sin 4A = 4 \sin A \cos A (1 - 2 \sin^2 A)$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

and so on.

INVERSE FUNCTIONS.

$$\sin^{-1} a + \sin^{-1} b = \sin^{-1} \{a \sqrt{(1-b^2)} + b \sqrt{(1-a^2)}\}$$

$$\cos^{-1} a + \cos^{-1} b = \cos^{-1} \{ab - \sqrt{(1-a^2)} \sqrt{(1-b^2)}\}$$

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1} \frac{a+b}{1-ab}$$

$$\cos^{-1} a = \frac{1}{2} \cos^{-1} (2a^2 - 1)$$

$$= \frac{1}{3} \cos^{-1} (4a^3 - 3a)$$

$$\sin^{-1} a = \frac{1}{2} \sin^{-1} 2a \sqrt{(1-a^2)}$$

$$= \frac{1}{3} \sin^{-1} (3a - 4a^3)$$

$$= \frac{1}{4} \sin^{-1} 4a(1-2a^2) \sqrt{(1-a^2)}$$

$$\tan^{-1} a = \frac{1}{2} \tan^{-1} \frac{2a}{1-a^2}$$

$$= \frac{1}{3} \tan^{-1} \frac{3a-a^3}{1-3a^2}$$

$$= \frac{1}{2} \sin^{-1} \frac{2a}{1+a^2}$$

$$= \frac{1}{2} \cos^{-1} \frac{1-a^2}{1+a^2}$$

- * (6) Differentiate, $\frac{2}{3}\sqrt{3}\tan^{-1}(2x+1)/\sqrt{3}$,
 $\sin^{-1}\{x\sqrt{1-a^2}+a\sqrt{1-x^2}\}$, $\sec^{-1}\frac{1}{2}(x+1/x)$,
 $2\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$, $\sin^{-1}\frac{\sqrt{1+x-x^2}}{\frac{1}{2}\sqrt{5}}$,
 $\cos^{-1}\sqrt{\frac{1-x+x^2}{3+3x+3x^2}}$, $\sin^{-1}\frac{2\sin x}{\sqrt{(5-6\cos x+5\cos^2 x)}}$.

* (7) Differentiate $f^{-1}x$ with respect to x .

27. The derivative of some of these inverse functions can be obtained almost as simply by the direct process, from the definition of § 3; thus

$$\begin{aligned}\frac{d \tan^{-1}x}{dx} &= \lim_{h \rightarrow 0} \frac{\tan^{-1}(x+h) - \tan^{-1}x}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \tan^{-1} \frac{x+h-x}{1+(x+h)x} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \tan^{-1} \frac{h}{1+x^2+xh} = \frac{1}{1+x^2};\end{aligned}$$

because $\lim_{z \rightarrow 0} (\tan^{-1}z)/z = \lim_{\theta \rightarrow 0} \theta/(\tan \theta) = 1$, when θ and $z=0$;

and here $\tan \theta = z = \frac{h}{1+x^2+xh}$.

* *Example.*—Determine in the same way the d.c. with respect to x of $\sin^{-1}x/a$, $\cos^{-1}x/a$, $\tan^{-1}x/a$, $\cot^{-1}x/a$, $\sec^{-1}x/a$, $\operatorname{cosec}^{-1}x/a$, $\operatorname{vers}^{-1}x/a$.

$$\begin{aligned}\left(\text{Thus } \frac{d \sin^{-1}x/a}{dx} = \lim_{h \rightarrow 0} \frac{\sin^{-1}(x+h)/a - \sin^{-1}x/a}{h}\right. \\ \left.= \lim_{h \rightarrow 0} \frac{\sin^{-1}z}{h} = \lim_{z \rightarrow 0} \frac{\sin^{-1}z}{z} \cdot \frac{z}{h},\right.\end{aligned}$$

where $z = \frac{x+h}{a} \sqrt{1 - \frac{x^2}{a^2}} - \frac{x}{a} \sqrt{1 - \left(\frac{x+h}{a}\right)^2}$;

and $z=0$, when $h=0$, so that $\lim_{z \rightarrow 0} (\sin^{-1}z)/z = 1$;

while $\lim_{h \rightarrow 0} \frac{z}{h} = \lim_{h \rightarrow 0} \frac{(x+h)\sqrt{(a^2-x^2)} - x\sqrt{a^2-(x+h)^2}}{a^2h} = 0$

(and rationalizing the numerator)

$$\begin{aligned}
&= \text{lt} \frac{(x+h)^2(a^2-x^2) - x^2\{a^2 - (x+h)^2\}}{a^2h[(x+h)\sqrt{(a^2-x^2)} + x\sqrt{\{a^2 - (x+h)^2\}}]} = \frac{0}{0} \\
&= \text{lt} \frac{2x+h}{(x+h)\sqrt{(a^2-x^2)} + x\sqrt{\{a^2 - (x+h)^2\}}} = \frac{1}{\sqrt{(a^2-x^2)}};
\end{aligned}$$

and thus $\frac{d \sin^{-1}x/a}{dx} = \frac{1}{\sqrt{(a^2-x^2)}}$, as before).

28. The Exponential Theorem. *Definition of the number e , and differentiation of $\log x$ and a^x .*

Before $d \log x/dx$ or da^x/dx can be found, the number e , called the *base of the natural or Napierian logarithms*, must be defined.

For, from the properties of logarithms,

$$\begin{aligned}
\frac{d \log_a x}{dx} &= \text{lt} \frac{\log_a(x+h) - \log_a x}{h} \\
&= \frac{1}{x} \text{lt} \frac{\log_a(1+h/x)}{h/x} = \frac{1}{x} \text{lt} \log_a \sqrt[1+z]{1+z},
\end{aligned}$$

when $z=0$, on putting $h/x=z$.

But now we must determine the value, when $z=0$, of $\sqrt[1+z]{1+z}$; or, on putting $z=1/m$, the value of $(1+1/m)^m = (1+0)^{1/0}$, when m is indefinitely great.

When m is an indefinitely large number, we may suppose it to be an integer; and now, by the Binomial Theorem, $\text{lt}(1+1/m)^m$, when $m=\infty$, $= (1+0)^{1/0}$,

$$\begin{aligned}
&= \text{lt} \left\{ 1 + m \frac{1}{m} + \frac{m(m-1)}{1 \cdot 2} \frac{1}{m^2} + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \frac{1}{m^3} + \dots \right\} \\
&= \text{lt} \left\{ 1 + \frac{1}{1} + \frac{1 - \frac{1}{m}}{1 \cdot 2} + \frac{\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)}{1 \cdot 2 \cdot 3} + \dots \right\} \\
&= 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots \frac{1}{n!} + \dots
\end{aligned}$$

(denoting the product $1 \cdot 2 \cdot 3 \dots n$, called *factorial n* , by $n!$)

.	=	1	= 1
		1	= 1/1 !
		·5	= 1/2 !
		·16666667	= 1/3 !
		·04166667	= 1/4 !
		·00833333	= 1/5 !
		·00138889	= 1/6 !
		·00019841	= 1/7 !
		·00002480	= 1/8 !
		·00000276	= 1/9 !
		·00000028	= 1/10 !
	
=		2·7182818...	

an incommensurable number, denoted generally by the letter e , and called the *base* of the natural logarithms.

29. Now replacing m by $1/z$, so that $z=0$ when $m=\infty$, then, when $z=0$, $\sqrt[1/z]{1+z}=e$; and

$$\text{lt. } \log_a \sqrt[1/z]{1+z} = \log_a e,$$

so that
$$\frac{d \log_a x}{dx} = \frac{1}{x} \log_a e,$$

and
$$\frac{d \log x}{dx} = \frac{1}{x},$$

and by the rule for the differentiation of a function of a function, $\frac{d \log r}{d\theta} = \frac{1}{r} \frac{dr}{d\theta}$ (§ 20), when the base is e .

Logarithms to base e are called *natural* logarithms; natural logarithms are intended in this subject when no base is indicated, and not *common* logarithms to base 10, as in ordinary numerical calculations; so also angles are always reckoned in circular measure, and not in degrees, minutes, or seconds: in this way extraneous factors are

avoided, although we must return to the other measurement when we wish to employ mathematical tables for numerical calculations; since logarithms are tabulated to base 10, and the circular functions are tabulated to degrees and minutes in the ordinary tables.

$$\text{Thus} \quad \frac{d \log_{10} x}{dx} = \frac{1}{x} \log_{10} e,$$

and $\log_{10} e = 0.43429448$, called M , the *modulus* of the common logarithms.

$$\text{Again} \quad \frac{da^x}{dx} = \text{lt} \frac{a^{x+h} - a^x}{h} = a^x \text{lt} \frac{a^h - 1}{h}.$$

$$\text{Now let} \quad a^h - 1 = z,$$

$$\text{therefore} \quad a^h = 1 + z,$$

$$\text{and} \quad h = \log_a(1 + z);$$

$$\text{also} \quad z = 0 \text{ when } h = 0.$$

$$\begin{aligned} \text{Therefore} \quad \frac{da^x}{dx} &= a^x \text{lt} \frac{z}{\log_a(1+z)} = \text{lt} \frac{a^x}{\log_a \sqrt[1]{1+z}} \\ &= \frac{a^x}{\log_a e} = a^x \log_e a. \end{aligned}$$

$$\text{Thus} \quad \frac{d10^x}{dx} = \frac{10^x}{M} = 10^x \times 2.30258509.$$

30. *The Exponential or Logarithmic Curve, and the Equiangular or Logarithmic Spiral.*

Logarithmic Coordinates.

The *exponential* or *logarithmic curve* is the graph of a^x ; so that its equation is $y = a^x$, or $x = \log_a y$.

It is the curve showing the rate at which a quantity grows in geometrical progression, or at compound interest, and is approximately the curve seen passing through the top of a row of organ pipes (fig. 11); equidistant ordinates being in geometrical progression.

Then $dy/dx = a^x \log a$, and $TM = ydx/dy = \log_a e$, so that the subtangent is constant in the logarithmic curve.

The rate per cent. at which $y = a^x$ grows per unit of x is $100 dy/ydx = 100 \log a$, which is constant; and denoting this rate per cent. by c , then $c = 100 \log a$, and $y = e^{cx/100}$.

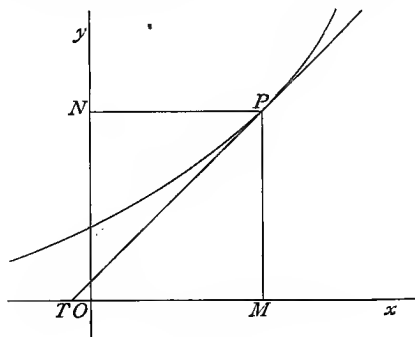


Fig. 11

Suppose for instance x denotes years; then in one year, y will have increased to $e^{c/100}$ times its value, which is at a rate of $100(e^{c/100} - 1)$ per cent. per annum.

The *equiangular* or *logarithmic spiral* is the graph in polar coordinates of a^θ , so that its equation is $r = a^\theta$ or $\theta = \log_a r$; it is called the equiangular spiral because the radial angle ϕ is constant.

For $\cot \phi = d \log r / d\theta = \log a$, a constant; and thus $r = ce^{\theta \cot \alpha}$ is the polar equation of an equiangular spiral, having a radial angle α (fig. 12 i.).

When the radial angle $\alpha = \frac{1}{2}\pi$, the equiangular spiral degenerates into a circle, $r = c$.

Equispaced vectors of this spiral increase in geometrical progression; but in $r = a\theta$, the spiral of Archimedes (fig. 12.ii.) the vectors increase in arithmetical progression.

In plotting graphically an empirical relation, of the form $y = ax^m$, between two variables x and y , say x the velocity of a projectile or ship and y the resistance of the air or water, a relation expressing the fact that $dy/y = m dx/x$, or that one per cent. increase in x gives m per cent.

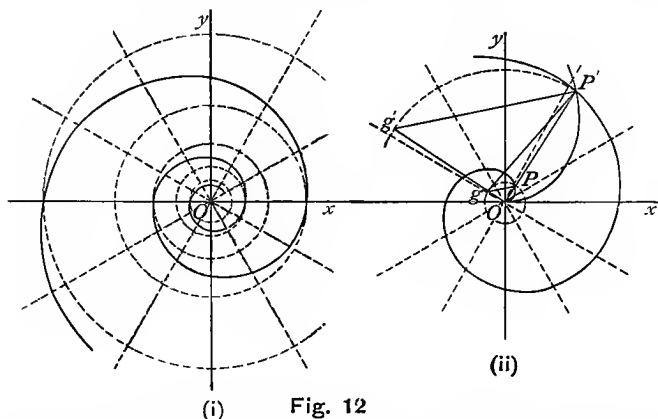


Fig. 12

increase in y , it is convenient sometimes to change to coordinates $\xi = \log x$ and $\eta = \log y$; and now the empirical relation becomes $\eta = m\xi + c$, so that the corresponding graph becomes a straight line.

*31. Logarithms.

The Exponential Theorem just employed forms part of Algebra, but for the sake of completeness the theorem has been established here.

Again the theorems of logarithms employed can easily be established as follows:—

Definition and principal properties of a Logarithm.—

When $a^m = p$, then m is called the logarithm of p to the base a : thus if $p = 10^m$, then m is the *common* logarithm of p , and if $p = e^m$, then m is the *natural* logarithm of p .

Now let $a^n = q$; then by the Theory of Indices, as explained in Algebra,

$$pq = a^m \times a^n = a^{m+n} \dots\dots\dots(1)$$

$$p/q = a^m/a^n = a^{m-n} \dots\dots\dots(2)$$

$$p^r = (a^m)^r = a^{rm} \dots\dots\dots(3)$$

$$\sqrt[r]{p} = (a^m)^{1/r} = a^{m/r} \dots\dots\dots(4)$$

Therefore the corresponding theorems in the logarithmic notation are

$$\log_a pq = m + n = \log_a p + \log_a q \dots\dots\dots(5)$$

$$\log p/q = m - n = \log p - \log q \dots\dots\dots(6)$$

$$\log p^r = rm = r \log p \dots\dots\dots(7)$$

$$\log \sqrt[r]{p} = m/r = (\log p)/r \dots\dots\dots(8)$$

Also if $e^b = a$, then $e = a^{1/b}$; and therefore

$$b = \log_e a, 1/b = \log_a e, \text{ so that } \log_e a \times \log_a e = 1;$$

and $p = a^m = e^{bm}$, so that $\log_a p = \log_a e \log_e p$.

Thus, $\log_{10} p = \log_{10} e \log_e p$, and $\log_{10} e = .4342945$, the modulus M , while $\log_e 10 = 2.3025851$.

These theorems have already been employed in establishing the preceding differentiations.

The *exponential* function a^x and the *logarithmic* function $\log_a x$ are functions inverse to each other; because, if $a^m = p$, then $m = \log_a p$; and $a^{\log_a x} = \log_a a^x = x$.

Starting with $m = \log_a p$, then p is sometimes denoted by $\log_a^{-1} m$, or by $\exp_a m$ (called *exponent* of m to base a) instead of by a^m , especially when m is a complicated mathematical expression, as thereby difficulties of printing are avoided.

The letters \exp must be considered an abbreviation of *exponent*; just as \log , usually employed, is an abbreviation for *logarithm*.

We have defined e as the $\text{lt}(1+1/m)^m$, when m is indefinitely great; and a similar expansion shows that

$$e^x = \text{lt}(1+1/m)^{mx} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

But suppose we reverse the procedure, and call this series $\exp x$; the series obtained by changing x into y will be $\exp y$; and then a straightforward multiplication of the two series shows that the product is the series

$$1 + \frac{x+y}{1!} + \frac{(x+y)^2}{2!} + \dots + \frac{(x+y)^n}{n!} + \dots;$$

so that $\exp(x+y) = \exp x \times \exp y$; and thus, generally,

$$\exp(x+y+z+\dots) = \exp x \times \exp y \times \exp z \times \dots$$

Now suppose there are n of these factors, and $x=y=z=\dots=1$; then since $\exp 1=e$, therefore $\exp n=e^n$, where n is a positive integer; and thence generally, by Induction, $\exp x=e^x$, for all values of x . (Cauchy, *Cours d'Analyse*; M. J. M. Hill, *Proc. Cam. Phil. Soc.* vol. v.)

Writing bx for x ,

$$e^{bx} = 1 + bx + \frac{b^2x^2}{2!} + \dots + \frac{b^nx^n}{n!} + \dots;$$

and putting $e^b=a$, so that $b=\log a$,

$$a^x = e^{bx} = e^{x \log a} = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \dots + \frac{x^n}{n!}(\log a)^n + \dots$$

Changing x into h , then

$$\frac{a^h - 1}{h} = \log a + \frac{h}{2!}(\log a)^2 + \dots + \frac{h^{n-1}}{n!}(\log a)^n + \dots,$$

reducing to its first term, $\log a$, when $h=0$, thus proving the lemma required in the differentiation of a^x .

Again $a^{x+h} = a^x a^h = a^x(1+h \log a + \dots)$, so that $a^x \log a$ is the coefficient of h in the expansion of a^{x+h} in ascending positive integral powers of h ; and is therefore, according to Lagrange's definition (§ 7), the derivative of a^x .

Putting $a = 1 + z$,
 then $(1 + z)^h = e^{h \log(1+z)}$

$$= 1 + h \log(1 + z) + \frac{h^2}{2!} \{\log(1 + z)\}^2 + \dots;$$

so that $\log(1 + z)$ is the coefficient of h in the expansion of $(1 + z)^h$ by the Binomial Theorem; and similarly $\{\log(1 + z)\}^2$ is twice the coefficient of h^2 , and so on; therefore

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots - \frac{(-1)^n z^n}{n} + \dots,$$

and $\log(1 - z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \dots - \frac{z^n}{n} - \dots;$

so that $\frac{1}{2} \log \frac{1+z}{1-z} = z + \frac{z^3}{3} + \frac{z^5}{5} + \dots + \frac{z^{2n+1}}{2n+1} + \dots$

Writing $1/n$ for z , then

$$\frac{1}{2} \log \frac{n+1}{n-1} = \frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \dots,$$

the series employed in calculating the natural logarithms.

Thus putting $n=2$ gives $\log 3$, $n=3$ gives $\log 2$, and thence $\log 4$, $\log 6$, $\log 8$, and $\log 9$; and putting $n=9$ gives $\log_e 10$, the reciprocal of which, $\log_{10} e$, is M , the modulus which converts natural into common logarithms.

We calculate logarithms to base e , but tabulate them to base 10; because numbers with the same series of digits and differing only in the position of the decimal point have the same *mantissa* or decimal part in the logarithm to base 10, and differ only in the *characteristic* or integral part, the value of which is easily written down.

Examples.

- (1) Calculate by logarithms the value of $(1 + 1/m)^m$, when m is put equal to 10, 100, 1000, 10000, ...; and show that these values tend to equality with e .

- (2) Deduce the differentiation of e^x from that of its inverse function $\log x$; and *vice versa*.
- *(3) Write down the series for $\{\log(1+z)\}^2$ in ascending powers of z .
- *(4) Prove algebraically that
- $$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)^2 - \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^2 = 1;$$
- $$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)^2 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^2 = 1;$$
- $$\exp\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} \dots\right) = 1 + x;$$
- $$\exp\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n} + \dots\right) = \frac{1}{1-x};$$
- $$\exp\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2n+1}}{2n+1} + \dots\right) = \sqrt{\frac{1+x}{1-x}}.$$
- *(5) Draw the graphs of 2^x , e^x , 10^x , $e^{-x}\cos x$, $e^{-x^2}\cos x$.
- *(6) Calculate, to base e , $\log 2$, $\log 3$, $\log 7$, $\log 10$; and thence their logarithms to base 10, to seven decimals.

32. Logarithmic Differentiation.

When the function to be differentiated is a single term consisting of factors raised to different powers, it is often simpler to take logarithms before differentiating; and then since (§ 29)

$$\frac{d \log y}{dx} = \frac{1}{y} \frac{dy}{dx},$$

therefore
$$\frac{dy}{dx} = y \frac{d \log y}{dx}.$$

Thus, if
$$y^2 = \frac{x^m}{(1+x)^n},$$

then
$$2 \log y = m \log x - n \log(1+x);$$

and differentiating with respect to x ,

$$\frac{2}{y} \frac{dy}{dx} = \frac{m}{x} - \frac{n}{1+x} = \frac{m + (m-n)x}{x(1+x)};$$

$$\frac{dy}{dx} = \frac{1}{2} \frac{m + (m-n)x}{(1+x)^{\frac{1}{2}n+1}} x^{\frac{1}{2}m-1};$$

this is called *logarithmic differentiation*.

Let $y = uvw\dots$,

the product of any number of functions u, v, w, \dots of x ;

then $\log y = \log u + \log v + \log w + \dots$

and differentiating this sum of functions,

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} + \dots$$

$$\frac{dy}{dx} = y \frac{d \log y}{dx}$$

$$= uvw\dots \left(\frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} + \dots \right),$$

the formula for the differentiation of a product (§ 12).

Again, let $y = u/v$,

$$\log y = \log u - \log v,$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} - \frac{1}{v} \frac{dv}{dx},$$

and

$$\frac{dy}{dx} = \left(\frac{du}{dx} v - u \frac{dv}{dx} \right) / v^2,$$

the formula for the differentiation of a quotient (§ 12).

Let $y = u^v$, where u and v are both functions of the independent variable x ; then

$$\log y = v \log u;$$

and differentiating this product with respect to x ,

$$\frac{1}{y} \frac{dy}{dx} = \frac{dv}{dx} \log u + \frac{v}{u} \frac{du}{dx},$$

$$\frac{du^v}{dx} = \frac{dy}{dx} = u^v \frac{dv}{dx} \log u + v u^{v-1} \frac{du}{dx}.$$

If u is constant, and v a function of x , then, by § 26,

$$\frac{du^v}{dx} = u^v \frac{dv}{dx} \log u;$$

and if v is constant and u a function of x , by §§ 3 and 11,

$$\frac{du^v}{dx} = v u^{v-1} \frac{du}{dx};$$

thus, when u and v are both functions of x , the differentiation of u^v is performed by supposing the functions to vary one at a time, and then adding the results.

Examples.—

- (1) Differentiate logarithmically with respect to x ,

$$\begin{aligned} & \sqrt{\frac{1-x}{1+x}}, \quad \frac{(x+1)^{\frac{1}{2}}(x+3)^{\frac{3}{2}}}{(x+2)^4}, \quad \frac{1}{(x-a)^m(x-b)^n}, \quad \frac{(\sin mx)^n}{(\cos nx)^m}, \\ & \sqrt{\frac{1+\sin x}{1-\sin x}}, \quad x^x, \quad e^{x^x}, \quad \sqrt{x}/x, \quad (1+1/x)^x, \quad \sqrt{x}/(1+x), \quad x^{\log x}, \\ & (\log x)^x, \quad (fx)^{fx}, \quad \log_x a. \end{aligned}$$

- (2) Prove the following differentiations,

$$(i.) \quad y = \log \sin x, \quad dy/dx = \cot x.$$

$$(ii.) \quad y = \log \sec x, \quad dy/dx = \tan x.$$

$$\begin{aligned} (iii.) \quad y &= \log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x) = \log(\sec x + \tan x), \\ &= \log \sqrt{\frac{1+\sin x}{1-\sin x}}, \quad dy/dx = \sec x. \end{aligned}$$

$$\begin{aligned} (iv.) \quad y &= \log \tan \tfrac{1}{2}x = \log(\operatorname{cosec} x - \cot x), \\ &= \log \sqrt{\frac{1-\cos x}{1+\cos x}}, \quad dy/dx = \operatorname{cosec} x. \end{aligned}$$

$$(v.) \quad y = \tfrac{1}{2} \sec^2 x - \log \sec x, \quad dy/dx = \tan^3 x.$$

$$\begin{aligned} (vi.) \quad y &= \tfrac{1}{2} \sec x \tan x + \tfrac{1}{2} \log(\sec x + \tan x), \\ & \quad dy/dx = \sec^3 x. \end{aligned}$$

$$(vii.) \quad y = e^x(x^2 - 2x + 2), \quad dy/dx = e^x x^2.$$

$$(viii.) \quad y = e^x(x^3 - 3x^2 + 6x - 6), \quad dy/dx = e^x x^3.$$

$$(ix.) \quad y = e^x \{x^n - nx^{n-1} + n(n-1)x^{n-2} - \dots\},$$

$$dy/dx = e^x x^n.$$

if n is a positive integer.

- (3) Prove that in the curve $y/a = \log \sec x/a$, or $e^{y/a} \cos x/a = 1$ (the *catenary of equal strength*),

$$\frac{ds}{dx} = \sec \frac{x}{a}, \quad \frac{ds}{dy} = \operatorname{cosec} \frac{x}{a}, \quad \text{and } x = a\psi.$$

- *(4) Draw the graphs of x^x , \sqrt{x} , $(1+1/x)^x$, $\sqrt[3]{1+x}$.

33. The Hyperbolic Functions.

Corresponding to the trigonometrical functions of the circle, defined in § 16, there are certain functions associated in a similar manner with the hyperbola, invented by Lambert (1768), called *hyperbolic functions*, which are of great use, and are defined here as follows:—

Taking the exponential function e^u and its reciprocal e^{-u} of any variable quantity u , then their half sum, $\frac{1}{2}(e^u + e^{-u})$, is called the *hyperbolic cosine* of u , and is denoted by $\cosh u$, and their half difference, $\frac{1}{2}(e^u - e^{-u})$ is called the *hyperbolic sine* of u , and is denoted by $\sinh u$.

The other hyperbolic functions of u are defined by analogy with the circular functions:

$\frac{\sinh u}{\cosh u}$	the hyperbolic tangent of u ,	denoted by $\tanh u$;
$\frac{\cosh u}{\sinh u}$	„	cotangent „ $\coth u$;
$1/\cosh u$	„	secant „ $\operatorname{sech} u$;
$1/\sinh u$	„	cosecant „ $\operatorname{cosech} u$;
$\cosh u - 1$	„	versed sine „ $\operatorname{versh} u$.

The reason for these names will appear hereafter; but the following formulas, which are easily established, show

that there is a Trigonometry of the hyperbolic functions exactly analogous to that of the circular functions; so that modern Trigonometry must be considered to include the properties of the circular and hyperbolic functions, which mutually assist and illustrate each other.

Thus

$$\begin{aligned}\cosh u + \sinh u &= e^u \quad \text{or } \exp u, \\ \cosh u - \sinh u &= e^{-u} \quad \text{or } \exp(-u); \\ \cosh^2 u - \sinh^2 u &= 1.\end{aligned}$$

Similarly

$$\begin{aligned}\tanh^2 u + \operatorname{sech}^2 u &= 1, \\ \coth^2 u - \operatorname{cosech}^2 u &= 1, \\ \sinh 2u &= 2 \sinh u \cosh u, \\ \cosh 2u &= \cosh^2 u + \sinh^2 u = 2 \cosh^2 u - 1 = 1 + 2 \sinh^2 u. \\ \sinh 2u &= \frac{2 \tanh u}{1 - \tanh^2 u}, \quad \cosh 2u = \frac{1 + \tanh^2 u}{1 - \tanh^2 u}, \\ \sinh(u + v) &= \sinh u \cosh v + \cosh u \sinh v, \\ \cosh(u + v) &= \cosh u \cosh v + \sinh u \sinh v, \\ \tanh(u + v) &= \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}.\end{aligned}$$

(These formulas are proved by noticing that

$$\begin{aligned}\cosh(u + v) &= \frac{1}{2} \exp(u + v) + \frac{1}{2} \exp(-u - v) \\ &= \frac{1}{2} \exp u \exp v + \frac{1}{2} \exp(-u) \exp(-v) \\ &= \frac{1}{2} (\cosh u + \sinh u)(\cosh v + \sinh v) \\ &\quad + \frac{1}{2} (\cosh u - \sinh u)(\cosh v - \sinh v) \\ &= \cosh u \cosh v + \sinh u \sinh v;\end{aligned}$$

and so on.)

$$\begin{aligned}\sinh u + \sinh v &= 2 \sinh \frac{1}{2}(u + v) \cosh \frac{1}{2}(u - v); \\ \sinh u - \sinh v &= 2 \cosh \frac{1}{2}(u + v) \sinh \frac{1}{2}(u - v); \\ \cosh u + \cosh v &= 2 \cosh \frac{1}{2}(u + v) \cosh \frac{1}{2}(u - v); \\ \cosh u - \cosh v &= 2 \sinh \frac{1}{2}(u + v) \sinh \frac{1}{2}(u - v); \\ \tanh u - \tanh v &= \sinh(u - v) \operatorname{sech} u \operatorname{sech} v.\end{aligned}$$

$OM = a \cos \theta$, $MP = a \sin \theta$, AR or $PT = a \tan \theta$, OR or $OT = a \sec \theta$, and $AM = a \text{ vers } \theta$.

Now if the ordinate TQ is erected of length equal to the tangent TP or AR , then the coordinates x and y of Q are given by $x = OT = a \sec \theta$, $y = TQ = a \tan \theta$; so that by the elimination of θ the equation of the locus of Q is

$$x^2 - y^2 = a^2,$$

and Q therefore describes a *rectangular hyperbola* AQ , while P describes the circle AP .

But since $\cosh^2 u - \sinh^2 u = 1$, we may also express the coordinates of Q more symmetrically by putting

$$OT = x = a \cosh u, \quad TQ = y = a \sinh u;$$

so that OT and TQ represent the hyperbolic cosine and sine of u in the hyperbola, just as OM and MP represent the circular cosine and sine of θ in the circle; hence the reason for these names.

Now $\cosh u = \sec \theta$, $\sinh u = \tan \theta$; and while $\frac{1}{2}a^2\theta$ is the area of the circular sector OAP , it will be found that $\frac{1}{2}a^2u$ is the area of the hyperbolic sector OAQ ; so that the analogy between θ and u is expressed through the areas of the circular and hyperbolic sectors, and not through the arcs or angles.

When θ and u are connected by this relation, then θ is called by Professor Cayley the *Gudermannian* of u , and sometimes also the *hyperbolic amplitude* of u , and is denoted by $\text{gd } u$, or $\text{amh } u$.

Conversely $\exp u = \cosh u + \sinh u = \sec \theta + \tan \theta$,

$$u = \text{gd}^{-1} \theta = \log(\sec \theta + \tan \theta) = \log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta),$$

by means of which u can be calculated as a function of θ , from the trigonometrical tables.

A Table in the Appendix, taken from Legendre's *Fonctions Elliptiques*, t. ii., gives the value of u for every degree in the angle whose c.m. is θ . When the hyperbolic functions for a certain value of u are required numerically, the corresponding value of θ is found by proportional parts; and then, by means of the tables,

$$\cosh u = \sec \theta, \sinh u = \tan \theta, \tanh u = \sin \theta, \dots$$

It may also be noticed that when $\cosh u = \sec \theta$, then

$$\frac{1 + \tanh^2 \frac{1}{2}u}{1 - \tanh^2 \frac{1}{2}u} = \frac{1 + \tan^2 \frac{1}{2}\theta}{1 - \tan^2 \frac{1}{2}\theta},$$

or

$$\tanh \frac{1}{2}u = \tan \frac{1}{2}\theta,$$

so that the line Otp , which bisects the angle or sector AOP , bisects also the hyperbolic sector AOQ , and therefore also the chord AQ ; and the tangents at A , P and Q intersect in t upon this line Otp .

Just as it was shown in § 16 that $\sin \theta < \theta < \tan \theta$; so from the fact that the triangle OAQ is greater than the hyperbolic sector OAQ , and that this is greater than the triangle OAU ; it follows that $\frac{1}{2}a^2 \sinh u$, $\frac{1}{2}a^2 u$, and $\frac{1}{2}a^2 \tanh u$ are in descending order of magnitude, or

$$\sinh u > u > \tanh u;$$

whence it follows, as in § 16, that when $u = 0$,

$$(\sinh u)/u = 1, \text{ and } (\tanh u)/u = 1.$$

**35. The Ellipse and Hyperbola compared.*

If the ordinates in fig. 13 are all reduced (or enlarged) in a constant ratio, say b/a , by orthogonal projection, or by throwing the shadow of fig. 13 on a plane parallel to Ox by parallel rays of light, then we obtain fig. 14, in which the curve AP is an ellipse, and the curve AQ is a hyperbola, no longer rectangular.

Now in the ellipse AP , $OM = a \cos \theta$, $MP = b \sin \theta$; while on the hyperbola AQ , $OT = a \cosh u$, $TQ = b \sinh u$,

so that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is the equation of the ellipse, and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

of the hyperbola.

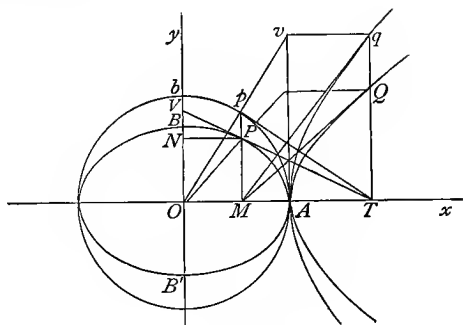


Fig.14

The angle θ is called the *excentric angle* of the point P , or in Astronomical language the *excentric anomaly*; while u may by analogy be called the *hyperbolic excentric anomaly* of the point Q on the hyperbola.

The elliptic sector will be b/a of the circular sector OAp of which it is the projection, and its area will therefore be $\frac{1}{2}a^2\theta \times b/a = \frac{1}{2}ab\theta$; and similarly the hyperbolic sector AOQ will be $\frac{1}{2}abu$ in area, or $\frac{1}{2}ab \log(x/a + y/b)$, since $\cosh u = x/a$, $\sinh u = y/b$; and therefore

$$\exp u = \frac{x}{a} + \frac{y}{b}, \quad u = \log \left(\frac{x}{a} + \frac{y}{b} \right).$$

Carrying out the properties of the ellipse and hyperbola on parallel lines by means of the excentric anomalies θ and u , we find:—

ELLIPSE.

Chord through (θ, ϕ) is

$$\frac{x}{a} \cos \frac{1}{2}(\theta + \phi) + \frac{y}{b} \sin \frac{1}{2}(\theta + \phi) \\ = \cos \frac{1}{2}(\theta - \phi).$$

Tangent at (θ) is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1.$$

Normal at (θ) is

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2.$$

Tangent at (θ, ϕ) meet at

$$x = a \frac{\cos \frac{1}{2}(\theta + \phi)}{\cos \frac{1}{2}(\theta - \phi)}, \quad y = b \frac{\sin \frac{1}{2}(\theta + \phi)}{\cos \frac{1}{2}(\theta - \phi)}.$$

Normals at (θ, ϕ) meet at

$$\frac{a^2 - b^2}{a} \frac{\cos \frac{1}{2}(\theta + \phi)}{\cos \frac{1}{2}(\theta - \phi)} \cos \theta \cos \phi, \\ - \frac{a^2 - b^2}{b} \frac{\sin \frac{1}{2}(\theta + \phi)}{\cos \frac{1}{2}(\theta - \phi)} \sin \theta \sin \phi.$$

HYPERBOLA.

Chord through (u, v) is

$$\frac{x}{a} \cosh \frac{1}{2}(u + v) - \frac{y}{b} \sinh \frac{1}{2}(u + v) \\ = \cosh \frac{1}{2}(u - v).$$

Tangent at (u) is

$$\frac{x}{a} \cosh u - \frac{y}{b} \sinh u = 1.$$

Normal at (u) is

$$ax \operatorname{sech} u + by \operatorname{cosech} u = a^2 + b^2.$$

Tangents at (u, v) meet at

$$x = a \frac{\cosh \frac{1}{2}(u + v)}{\cosh \frac{1}{2}(u - v)}, \quad y = b \frac{\sinh \frac{1}{2}(u + v)}{\cosh \frac{1}{2}(u - v)},$$

Normals at (u, v) meet at

$$\frac{a^2 + b^2}{a} \frac{\cosh \frac{1}{2}(u + v)}{\cosh \frac{1}{2}(u - v)} \cosh u \cosh v, \\ - \frac{a^2 + b^2}{b} \frac{\sinh \frac{1}{2}(u + v)}{\cosh \frac{1}{2}(u - v)} \sinh u \sinh v.$$

Generally in the curve $(x/a)^n \pm (y/b)^n = 1$, we may put $(x/a)^n = \cos^2 \theta$ or $\cosh^2 u$, $(y/b)^n = \sin^2 \theta$ or $\sinh^2 u$; and then the equation of the tangent is

$$x/a (\cos \theta)^{2-2/n} + (y/b) (\sin \theta)^{2-2/n} = 1,$$

$$\text{or} \quad (x/a) (\cosh u)^{2-2/n} - (y/b) (\sinh u)^{2-2/n} = 1.$$

36. *The Catenary.*

To illustrate the practical use of the hyperbolic functions, an instructive curve to take is the *catenary*, the curve in which a uniform chain hangs; the equation is

$$y/a = \cosh x/a;$$

so that the catenary is the graph of $a \cosh x/a$, and the ordinate of the catenary is the arithmetic mean of the ordinates of the two exponential curves

$$y/a = e^{x/a}, \text{ and } y/a = e^{-x/a}.$$

The normal $PG = y \sec \psi = y^2/\alpha$.

By a change of origin from O to A the equation of the catenary becomes

$$y + \alpha = \alpha \cosh x/\alpha$$

or $y/2\alpha = \sinh^2 x/2\alpha$; or $y/b = \sinh^2 x/b$, if $b = 2\alpha$;

and now $(y + \alpha)^2 = s^2 + \alpha^2$, or $s^2 = y^2 + 2\alpha y = y^2 + by$.

Suppose for example a steel telegraph wire 5000 feet long is stretched between two points P, P' at the same level, so that the versed sine or dip NA in the middle is 500 feet; then $y = 500$, $s = 2500$, and therefore $\alpha = 6000$; so that the tension at P or P' is the weight of 6500 feet of wire; and taking the specific gravity of steel as 8, this would give a tensile stress of about 10 tons per sq. inch. Then $2x$, the distance between P and P' , will be 4865 feet.

When the wire is screwed up tight so that the dip y is small, then s and x are very nearly equal, and

$$\alpha = \frac{1}{2}s^2/y - \frac{1}{2}y \approx \frac{1}{2}s^2/y \approx \frac{1}{2}x^2/y;$$

so that if telegraph wire sags y feet in the middle between posts l feet apart, the wire is screwed up to a tension equal to the weight of about $\frac{1}{8}l^2/y$ feet of wire.

* If an endless chain, of length $2l$ or $4s$, is suspended over smooth pulleys at P and P' (fig. 15), then when P and P' are almost as far apart as possible, the two parts of the chain coincide in a single catenary; but when P and P' are moderately close, the chain hangs in two distinct festoons, catenaries having the same directrix, since the tension is unchanged in passing round the smooth pulleys at P and P' . Also, if the festoons are partly supported by smooth inclined planes, the various catenaries will have the same directrix.

Any catenary between the two festoons will have a higher directrix, and beyond them a lower directrix.

Drawing the two common tangents to the two festoons, they will meet in O , since the catenaries of the two festoons are similar curves, and O is a centre of similitude.

Therefore the tangents at P, P' of the upper festoon intersect above the directrix, and below the directrix for the tangents at P, P' of the lower festoon.

Now, let P and P' be drawn apart; the directrix of the two festoons will rise until they coalesce, when the tangents at P and P' will intersect in O ; afterwards the directrix will descend again.

In the separating case then, the tangents at P and P' intersect in O ; and then

$$\tan \psi = \frac{y}{x} = \frac{dy}{dx}, \text{ or } \frac{a}{x} \cosh \frac{x}{a} = \sinh \frac{x}{a},$$

$$\text{or } \operatorname{cosec} \psi = \coth x/a = x/a, \quad \frac{x}{a} = \frac{1}{2} \log \frac{x+a}{x-a},$$

from which transcendental equation we find, by aid of the Table in the Appendix, the approximate root $x/a \approx 1.2$; and then $y/x = s/a \approx 1.5$, $s/x \approx 1.25$, $x/s \approx .8$; and ψ is the c.m. of $56^\circ 30'$ about.

The profile of the catenary is also seen in the surface of revolution, called the *catenoid*, formed by a soap bubble film, adhering to a circular wire which is raised gently in a horizontal position from the surface of soapy water.

The same conditions give the stability of this capillary film (fig. 16); so long as the tangent cone along PP' has its vertex below the surface of the water, the film is stable; but the film always breaks when the vertex reaches the surface, and then x , the height of the wire

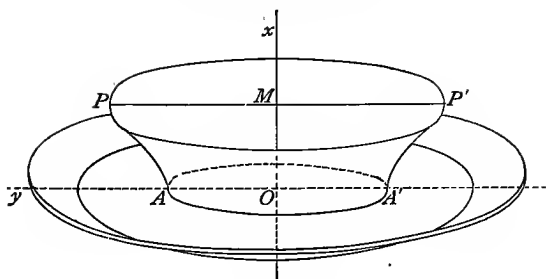


Fig.16

PP' above the surface, and $2y$ its diameter are connected by

$$2y/x \approx 3, \quad x/2y \approx \frac{1}{3}.$$

(Maxwell, *Capillary Action*, Encyclopædia Britannica.)

If a thread, wrapped upon the catenary AP , is cut at A and then unwrapped, the end will describe the curve AQ , and QP will be the normal of this curve at P .

Since QM , the tangent of this curve, is of constant length a , the curve Q is that which would be described by a body on a rough plane if drawn very slowly by the point M , moving in Ox , by a thread or chain MQ of constant length a ; hence the curve AQ is called the *Tractrix*.

Examples.

(1) Prove the following differentiations :

- | | |
|--|-----------------------|
| (i.) $y = \frac{1}{4} \sinh 2x + \frac{1}{2}x$, | $dy/dx = \cosh^2 x$. |
| (ii.) $y = \frac{1}{4} \sinh 2x - \frac{1}{2}x$, | $dy/dx = \sinh^2 x$. |
| (iii.) $y = \frac{1}{3} \cosh^3 x - \cosh x$, | $dy/dx = \sinh^3 x$. |
| (iv.) $y = \frac{1}{3} \sinh 4x + \frac{1}{4} \sinh 2x + \frac{3}{8}x$, | $dy/dx = \cosh^4 x$. |
| (v.) $y = \log \cosh x$, | $dy/dx = \tanh x$. |
| (vi.) $y = \log \sinh x$, | $dy/dx = \coth x$. |
| (vii.) $y = x - \tanh x$, | $dy/dx = \tanh^2 x$. |

$$(viii.) \quad y = \log \cosh x - \frac{1}{2} \tanh^2 x, \quad dy/dx = \tanh^3 x.$$

$$(ix.) \quad y = x - \tanh x - \frac{1}{3} \tanh^3 x, \quad dy/dx = \tanh^4 x.$$

$$(x.) \quad y = \cos^{-1} \operatorname{sech} x, \text{ or } \sin^{-1} \tanh x, \text{ or } \tan^{-1} \sinh x, \\ \text{or } 2 \tan^{-1} \tanh \frac{1}{2} x, \quad dy/dx = \operatorname{sech} x.$$

$$*(xi.) \quad y = \log \tanh \frac{1}{2} x, \text{ or } -\cosh^{-1} \coth x, \text{ or } -\sinh^{-1} \operatorname{cosech} x, \\ \text{or } -\tanh^{-1} \operatorname{sech} x, \quad dy/dx = \operatorname{cosech} x.$$

$$(xii.) \quad y = \frac{1}{2} \log \coth \frac{1}{2} x - \frac{1}{2} \operatorname{cosech} x \coth x, \\ dy/dx = \operatorname{cosech}^3 x.$$

$$(xiii.) \quad y = \tanh x - \frac{1}{3} \tanh^3 x, \quad dy/dx = \operatorname{sech}^4 x.$$

(2) Prove that the equation of the *catenary of equal strength*, $y/a = \log \sec x/a$, may also be written $\tanh \frac{1}{2} y/a = \tan^2 \frac{1}{2} x/a$.

(3) Prove that the equation of the curve

$$e^{(x+y)/a} = e^{x/a} + e^{y/a} + 1,$$

may also be written

$$\sinh x/a \sinh y/a = 1, \text{ or } y/a = \log \coth \frac{1}{2} x/a,$$

and that $dy/dx = \operatorname{cosech} x/a$, $ds/dx = \coth x/a$.

*37. The Inverse Hyperbolic Functions.

To preserve a complete analogy it is convenient to have the *inverse hyperbolic functions* as well as the inverse circular functions of § 25, the method of notation being exactly the same; thus if $\cosh u = x$, then $u = \cosh^{-1} x$, and so on.

The inverse hyperbolic functions are however only variations of the logarithmic function; thus if

$$\cosh u = x, \text{ then } \sinh u = \sqrt{(x^2 - 1)},$$

$$\text{and} \quad \exp u = \cosh u + \sinh u = x + \sqrt{(x^2 - 1)},$$

$$\text{so that} \quad u = \cosh^{-1} x = \log \{x + \sqrt{(x^2 - 1)}\};$$

$$\text{similarly} \quad \sinh^{-1} x = \log \{\sqrt{(1 + x^2)} + x\}.$$

Again if $\tanh u = x$,

$$\text{then } \frac{e^u - e^{-u}}{e^u + e^{-u}} = x, \text{ or } e^{2u} = \frac{1+x}{1-x},$$

$$\text{and } u = \tanh^{-1}x = \frac{1}{2} \log \frac{1+x}{1-x};$$

$$\text{similarly } \coth^{-1}x = \frac{1}{2} \log \frac{x+1}{x-1}.$$

The use of the inverse hyperbolic functions will be apparent when we compare their derivatives with those of the inverse circular functions.

Thus let $\cosh^{-1}x/a = u$; then $x = a \cosh u$,

$$dx/du = a \sinh u = a\sqrt{(\cosh^2 u - 1)} = \sqrt{(x^2 - a^2)},$$

$$\text{and } \frac{du}{dx} = \frac{d \cosh^{-1}x/a}{dx} = \frac{1}{\sqrt{(x^2 - a^2)}}.$$

$$\text{Similarly } \frac{d \sinh^{-1}x/a}{dx} = \frac{1}{\sqrt{(a^2 + x^2)}},$$

$$\text{while } \frac{d \sin^{-1}x/a}{dx} = -\frac{d \cos^{-1}x/a}{dx} = \frac{1}{\sqrt{(a^2 - x^2)}}.$$

Again,

$$\frac{d \tanh^{-1}x/a}{dx} = \frac{a}{a^2 - x^2}, (x < a); \quad \frac{d \coth^{-1}x/a}{dx} = \frac{a}{a^2 - x^2}, (x > a),$$

$$\text{while } \frac{d \tan^{-1}x/a}{dx} = -\frac{d \cot^{-1}x/a}{dx} = \frac{a}{a^2 + x^2}.$$

It will be instructive for the student to make a table of the principal formulas of the circular and hyperbolic functions in parallel columns, to show the correspondence; this will be found to assist the memory in recollecting the formulas.

Examples.

- (1) Construct a Table giving any direct hyperbolic function in terms of any other; and differentiate them as a verification.
- (2) Construct a similar Table for the inverse hyperbolic functions; and differentiate them.
- (3) Prove that

$$\begin{aligned}\cosh^{-1}x + \cosh^{-1}y &= \cosh^{-1}\{xy + \sqrt{(x^2-1)}\sqrt{(y^2-1)}\} \\ &= \sinh^{-1}\{x\sqrt{(y^2-1)} + y\sqrt{(x^2-1)}\}\end{aligned}$$

$$\begin{aligned}\sinh^{-1}x + \sinh^{-1}y &= \sinh^{-1}\{x\sqrt{(1+y^2)} + y\sqrt{(1+x^2)}\} \\ &= \cosh^{-1}\{\sqrt{(x^2+1)}\sqrt{(y^2+1)} + xy\}\end{aligned}$$

$$\tanh^{-1}x + \tanh^{-1}y = \tanh^{-1} \frac{x+y}{1+xy}.$$

- (4) Prove that if

$$\begin{aligned}y = \cosh^{-1}x &= \frac{1}{2} \cosh^{-1}(2x^2-1) = \frac{1}{2} \sinh^{-1}2x\sqrt{(x^2-1)} \\ &= \frac{1}{3} \cosh^{-1}(4x^3-3x), \dots\end{aligned}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(x^2-1)}}.$$

- (5) Prove that if

$$\begin{aligned}y = \sinh^{-1}x &= \frac{1}{2} \sinh^{-1}2x\sqrt{(1+x^2)} = \frac{1}{2} \cosh^{-1}(1+2x^2) \\ &= \frac{1}{3} \sinh^{-1}(3x+4x^3), \dots\end{aligned}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1+x^2)}}.$$

- (6) $y = \tanh^{-1}x = \frac{1}{2} \tanh^{-1} \frac{2x}{1+x^2} = \frac{1}{2} \sinh^{-1} \frac{2x}{1-x^2}$
- $$= \frac{1}{2} \cosh^{-1} \frac{1+x^2}{1-x^2} = \frac{1}{3} \tanh^{-1} \frac{3x+x^3}{1+3x^2}; \dots$$

$$\frac{dy}{dx} = \frac{1}{1-x^2}.$$

(7) Prove that if

$$\begin{aligned}\theta = \operatorname{gd} u &= \cos^{-1} \operatorname{sech} u = \sin^{-1} \tanh u = \tan^{-1} \sinh u \\ &= 2 \tan^{-1} \tanh \frac{1}{2} u = 2 \tan^{-1} e^u - \frac{1}{2} \pi; \\ \frac{d\theta}{du} &= \operatorname{sech} u.\end{aligned}$$

(8) Prove that if

$$\begin{aligned}u = \operatorname{gd}^{-1} \theta &= \cosh^{-1} \sec \theta = \sinh^{-1} \tan \theta = \tanh^{-1} \sin \theta \\ &= 2 \tanh^{-1} \tan \frac{1}{2} \theta, \\ \frac{du}{d\theta} &= \sec \theta.\end{aligned}$$

(9) Prove that

$$\begin{aligned}\cosh \log(\sec \theta + \tan \theta) &= \sec \theta. \\ \sinh \log(\sec \theta + \tan \theta) &= \tan \theta. \\ \tanh \log(\sec \theta + \tan \theta) &= \sin \theta, \text{ etc.}\end{aligned}$$

(10) In figure 13, prove that

- (i.) $\angle A'AP + \angle AA'P = \frac{1}{2}\pi$; (ii.) $\angle A'AQ - \angle AA'Q = \frac{1}{2}\pi$.
(iii.) $A'P$ and AP' intersect in Q , PMP' being the chord of the circle.

Prove, geometrically, $\sec \theta \pm \tan \theta = \tan(\frac{1}{4}\pi \pm \frac{1}{2}\theta)$.

$$(11) \quad y = \cosh^{-1} \frac{\sqrt{(x^2 + x + 1)}}{\frac{1}{2}\sqrt{3(x+1)}}, \quad \frac{dy}{dx} = \frac{1}{(x+1)\sqrt{(x^2 + x + 1)}}.$$

$$(12) \quad y = \sinh^{-1} \frac{\sqrt{(x^2 + x - 1)}}{\frac{1}{2}\sqrt{5(x-1)}}, \quad \frac{dy}{dx} = \frac{1}{(x-1)\sqrt{(x^2 + x - 1)}}.$$

* *General Examples of Differentiation.*

In the following examples y is given as a function of x , and it is required to determine dy/dx , according to the rules explained in this chapter. These examples are arranged to serve as guides for the results of Integration, required hereafter.

$$(1) \ y = \log \sqrt{\frac{1+x}{1-x}}, \text{ or } \log \sqrt{\frac{x+1}{x-1}}, \quad \frac{dy}{dx} = \frac{1}{1-x^2}.$$

$$(2) \ y = \frac{1}{\sqrt{(ac-b^2)}} \tan^{-1} \frac{ax+b}{\sqrt{(ac-b^2)}},$$

$$\text{or} \quad \frac{1}{\sqrt{(b^2-ac)}} \log \sqrt{\frac{ax+b-\sqrt{(b^2-ac)}}{ax+b+\sqrt{(b^2-ac)}}},$$

$$\frac{dy}{dx} = \frac{1}{ax^2+2bx+c};$$

$$\text{and } y = \frac{P}{2a} \log(ax^2+2bx+c) + \frac{Qa-Pb}{a\sqrt{(ac-b^2)}} \tan^{-1} \frac{ax+b}{\sqrt{(ac-b^2)}},$$

$$\frac{dy}{dx} = \frac{Px+Q}{ax^2+2bx+c}.$$

$$(3) \ y = \frac{1}{6} \log \frac{(x+1)^3}{x^3+1} + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}}, \quad \frac{dy}{dx} = \frac{1}{x^3+1}.$$

$$(4) \ y = A \log(x-1) + B \log(x^2+x+1) \\ + C \tan^{-1} \frac{1}{\sqrt{3}} \sqrt{3(2x+1)};$$

$$\text{determine } A, B, C, \text{ when } \frac{dy}{dx} = \frac{1}{x^3-1}, \text{ or } x, \text{ or } x^2.$$

$$(5) \ y = \frac{1}{2} \tanh^{-1} x + \frac{1}{2} \tan^{-1} x$$

$$= \log \sqrt{\frac{1+x}{1-x}} + \frac{1}{2} \tan^{-1} x, \quad \frac{dy}{dx} = \frac{1}{1-x^4}.$$

$$(6) \ y = \frac{1}{2} \cot^{-1} x - \frac{1}{2} \coth^{-1} x$$

$$= \frac{1}{2} \cot^{-1} x - \log \sqrt{\frac{x+1}{x-1}}, \quad \frac{dy}{dx} = \frac{1}{x^4-1}.$$

$$(7) \quad y = A \log(x-1) + B \log(x+1) + C \log(x^2+1) + D \tan^{-1}x;$$

determine A, B, C, D when $\frac{dy}{dx} = \frac{x, \text{ or } x^2, \text{ or } x^3}{1-x^4}$.

$$(8) \quad y = \frac{1}{2} \sqrt{2} \log \sqrt{\frac{x^2 + \sqrt{2x+1}}{x^2 - \sqrt{2x+1}}} - \frac{1}{4} \sqrt{2} \tan^{-1} \frac{\sqrt{2x}}{x^2-1}$$

$$= \frac{1}{4} \sqrt{2} \tanh^{-1} \frac{\sqrt{2x}}{x^2+1} - \frac{1}{4} \sqrt{2} \tan^{-1} \frac{\sqrt{2x}}{x^2-1}, \quad \frac{dy}{dx} = \frac{1}{x^4+1}.$$

$$(9) \quad y = A \log(x^2 - \sqrt{2x+1}) + B \log(x^2 + \sqrt{2x+1})$$

$$+ C \tan^{-1}(\sqrt{2x-1}) + D \tan^{-1}(\sqrt{2x+1});$$

determine A, B, C, D , when $\frac{dy}{dx} = \frac{x, \text{ or } x^2, \text{ or } x^3}{x^4+1}$.

$$(10) \quad y = 2 \cos^{-1} \sqrt{\frac{\alpha-x}{\alpha-\beta}} = 2 \sin^{-1} \sqrt{\frac{x-\beta}{\alpha-\beta}} = 2 \tan^{-1} \sqrt{\frac{x-\beta}{\alpha-x}}$$

$$= \sin^{-1} \frac{\sqrt{(\alpha-x)(x-\beta)}}{\frac{1}{2}(\alpha-\beta)}; \quad \frac{dy}{dx} = \frac{1}{\sqrt{(\alpha-x)(x-\beta)}}.$$

$$(11) \quad y = 2 \sinh^{-1} \sqrt{\frac{x-\alpha}{\alpha-\beta}} = 2 \cosh^{-1} \sqrt{\frac{x-\beta}{\alpha-\beta}} = 2 \tanh^{-1} \sqrt{\frac{x-\alpha}{x-\beta}}$$

$$= \sinh^{-1} \frac{\sqrt{(x-\alpha)(x-\beta)}}{\frac{1}{2}(\alpha-\beta)}$$

$$= 2 \log \{ \sqrt{(x-\alpha)} + \sqrt{(x-\beta)} \} - \log(\alpha-\beta);$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(x-\alpha)(x-\beta)}}.$$

$$(12) \quad y = 2 \cosh^{-1} \sqrt{\frac{\alpha-x}{\alpha-\beta}} = 2 \sinh^{-1} \sqrt{\frac{\beta-x}{\alpha-\beta}} = 2 \tanh^{-1} \sqrt{\frac{\beta-x}{\alpha-x}}$$

$$= \sinh^{-1} \frac{\sqrt{(\alpha-x)(\beta-x)}}{\frac{1}{2}(\alpha-\beta)}$$

$$= 2 \log \{ \sqrt{(\alpha-x)} + \sqrt{(\beta-x)} \} - \log(\alpha-\beta)$$

$$= \log \frac{\sqrt{(\alpha-x)} + \sqrt{(\beta-x)}}{\sqrt{(\alpha-x)} - \sqrt{(\beta-x)}}; \quad \frac{dy}{dx} = \frac{-1}{\sqrt{(\alpha-x)(\beta-x)}}.$$

$$\begin{aligned}
 (13) \quad \frac{dy}{dx} &= \frac{1}{\sqrt{(ax^2 + 2bx + c)}}, \text{ if} \\
 y &= \frac{1}{\sqrt{(-a)}} \sin^{-1} \frac{\sqrt{(-a)} \sqrt{(ax^2 + 2bx + c)}}{\sqrt{(b^2 - ac)}} \\
 &= \frac{1}{\sqrt{(-a)}} \cos^{-1} \frac{ax + b}{\sqrt{(b^2 - ac)}} \\
 \text{or} \quad \frac{1}{\sqrt{a}} \sinh^{-1} \frac{\sqrt{a} \sqrt{(ax^2 + 2bx + c)}}{\sqrt{(b^2 - ac)}} \\
 &= \frac{1}{\sqrt{a}} \cosh^{-1} \frac{ax + b}{\sqrt{(b^2 - ac)}} \\
 \text{or} \quad \frac{1}{\sqrt{a}} \cosh^{-1} \frac{\sqrt{a} \sqrt{(ax^2 + 2bx + c)}}{\sqrt{(ac - b^2)}} \\
 &= \frac{1}{\sqrt{a}} \sinh^{-1} \frac{ax + b}{\sqrt{(ac - b^2)}};
 \end{aligned}$$

Determine the degenerate form when $a = 0$.

$$\begin{aligned}
 (14) \quad \frac{dy}{dx} &= \frac{1}{(b-x)\sqrt{(x-a)}}, \text{ if} \\
 y &= \frac{2}{\sqrt{(a-b)}} \cos^{-1} \sqrt{\frac{x-a}{x-b}}, \\
 \text{or} \quad \frac{2}{\sqrt{(b-a)}} \cosh^{-1} \sqrt{\frac{x-a}{x-b}}, \\
 \text{or} \quad \frac{2}{\sqrt{(b-a)}} \sinh^{-1} \sqrt{\frac{x-a}{b-x}}; \\
 (15) \quad \frac{dy}{dx} &= \frac{1}{(b-x)\sqrt{(a-x)}}, \text{ if} \\
 y &= \frac{2}{\sqrt{(b-a)}} \cos^{-1} \sqrt{\frac{a-x}{b-x}}, \\
 \text{or} \quad \frac{2}{\sqrt{(a-b)}} \cosh^{-1} \sqrt{\frac{a-x}{b-x}}, \\
 \text{or} \quad \frac{2}{\sqrt{(a-b)}} \sinh^{-1} \sqrt{\frac{a-x}{x-b}};
 \end{aligned}$$

$$(16) \quad \frac{dy}{dx} = \frac{1}{(x-\gamma)\sqrt{(x-a)(x-\beta)}}, \text{ if}$$

$$y = \frac{1}{\sqrt{(a-\gamma)(\gamma-\beta)}} \sin^{-1} \frac{\sqrt{(a-\gamma)(\gamma-\beta)}\sqrt{(x-a)(x-\beta)}}{\frac{1}{2}(a-\beta)(x-\gamma)},$$

$$\text{or } \frac{1}{\sqrt{(\gamma-a)(\gamma-\beta)}} \sinh^{-1} \frac{\sqrt{(\gamma-a)(\gamma-\beta)}\sqrt{(x-a)(x-\beta)}}{\frac{1}{2}(a-\beta)(x-\gamma)}.$$

$$(17) \quad \frac{dy}{dx} = \frac{1}{(x-\gamma)\sqrt{(a-x)(x-\beta)}}, \text{ if}$$

$$y = \frac{1}{\sqrt{(\gamma-a)(\gamma-\beta)}} \sin^{-1} \frac{\sqrt{(\gamma-a)(\gamma-\beta)}\sqrt{(a-x)(x-\beta)}}{\frac{1}{2}(a-\beta)(x-\gamma)},$$

$$\text{or } \frac{1}{\sqrt{(a-\gamma)(\gamma-\beta)}} \sinh^{-1} \frac{\sqrt{(a-\gamma)(\gamma-\beta)}\sqrt{(a-x)(x-\beta)}}{\frac{1}{2}(a-\beta)(x-\gamma)}.$$

Determine the degenerate forms when $\gamma = a$ or β .

(18) Denoting $ax^2 + 2bx + c$ by R , prove that

$$\frac{dy}{dx} = \frac{1}{(x-p)\sqrt{R}} = \frac{1}{(x-p)\sqrt{(ax^2 + 2bx + c)}}, \text{ if}$$

$$y = \frac{1}{\sqrt{(-ap^2 - 2bp - c)}} \sin^{-1} \frac{\sqrt{(-ap^2 - 2bp - c)}\sqrt{R}}{\sqrt{(b^2 - ac)(x-p)}}$$

$$= \frac{1}{\sqrt{(-ap^2 - 2bp - c)}} \cos^{-1} \frac{(ap+b)x + bp + c}{\sqrt{(b^2 - ac)(x-p)}},$$

$$\text{or } \frac{1}{\sqrt{(ap^2 + 2bp + c)}} \sinh^{-1} \frac{\sqrt{(ap^2 + 2bp + c)}\sqrt{R}}{\sqrt{(b^2 - ac)(x-p)}}$$

$$= \frac{1}{\sqrt{(ap^2 + 2bp + c)}} \cosh^{-1} \frac{(ap+b)x + bp + c}{\sqrt{(b^2 - ac)(x-p)}},$$

$$\text{or } \frac{1}{\sqrt{(ap^2 + 2bp + c)}} \cosh^{-1} \frac{\sqrt{(ap^2 + 2bp + c)}\sqrt{R}}{\sqrt{(ac - b^2)(x-p)}}$$

$$= \frac{1}{\sqrt{(ap^2 + 2bp + c)}} \sinh^{-1} \frac{(ap+b)x + bp + c}{\sqrt{(ac - b^2)(x-p)}}.$$

$$\begin{aligned}
 (19) \quad y &= \frac{1}{\sqrt{C}\sqrt{Ac-aC}} \cos^{-1} \sqrt{\left(\frac{C}{c} \frac{ax^2+c}{Ax^2+C} \right)}, \\
 \text{or} \quad & \frac{1}{\sqrt{C}\sqrt{aC-Ac}} \cosh^{-1} \sqrt{\left(\frac{C}{c} \frac{ax^2+c}{Ax^2+C} \right)}, \\
 \text{or} \quad & \frac{1}{\sqrt{C}\sqrt{aC-Ac}} \sinh^{-1} \sqrt{\left(-\frac{C}{c} \frac{ax^2+c}{Ax^2+C} \right)}; \\
 \frac{dy}{dx} &= \frac{1}{(Ax^2+C)\sqrt{(ax^2+c)}}.
 \end{aligned}$$

$$\begin{aligned}
 (20) \quad y &= e^{ax+b} \cos(px+q), \\
 \frac{dy}{dx} &= a \sec ae^{ax+b} \cos(px+q+a), \text{ where } \tan a = p/a.
 \end{aligned}$$

$$\begin{aligned}
 (21) \quad y &= m \sin mx \cosh nx + n \cos mx \sinh nx, \\
 \frac{dy}{dx} &= (m^2 + n^2) \cos mx \cosh nx,
 \end{aligned}$$

$$(22) \quad y = \frac{1}{2}x\sqrt{(a^2-x^2)} + \frac{1}{2}a^2 \sin^{-1}x/a, \quad \frac{dy}{dx} = \sqrt{(a^2-x^2)}.$$

$$\begin{aligned}
 (23) \quad y &= \frac{1}{2}a^2 \text{vers}^{-1}x/a - \frac{1}{2}(a-x)\sqrt{(2ax-x^2)}, \\
 \frac{dy}{dx} &= \sqrt{(2ax-x^2)}.
 \end{aligned}$$

$$(24) \quad y = \frac{1}{2}x\sqrt{(x^2-a^2)} - \frac{1}{2}a^2 \cosh^{-1}x/a, \quad \frac{dy}{dx} = \sqrt{(x^2-a^2)}.$$

$$(25) \quad y = \frac{1}{2}x\sqrt{(a^2+x^2)} + \frac{1}{2}a^2 \sinh^{-1}x/a, \quad \frac{dy}{dx} = \sqrt{(a^2+x^2)}.$$

$$\begin{aligned}
 (26) \quad y &= \frac{1}{2}(a+x)\sqrt{(2ax+x^2)} - \frac{1}{2}a^2 \cosh^{-1}(1+x/a), \\
 \frac{dy}{dx} &= \sqrt{(2ax+x^2)}.
 \end{aligned}$$

$$\begin{aligned}
 (27) \quad y &= \frac{1}{2}(x-a)\sqrt{(x^2-2ax)} - \frac{1}{2}a^2 \sinh^{-1}\sqrt{(x^2-2ax)}/a, \\
 \frac{dy}{dx} &= \sqrt{(x^2-2ax)}.
 \end{aligned}$$

(28) Denoting $ax^2 + 2bx + c$ by R , prove that $\frac{dy}{dx} = \sqrt{R}$, if

$$y = \frac{\frac{1}{2}(ax+b)\sqrt{R}}{a} - \frac{1}{2} \frac{b^2-ac}{a} \frac{1}{\sqrt{(-a)}} \sin^{-1} \frac{\sqrt{(-a)}\sqrt{R}}{\sqrt{(b^2-ac)}},$$

or $\frac{1}{\sqrt{a}} \sinh^{-1} \frac{\sqrt{a}\sqrt{R}}{\sqrt{(b^2-ac)}}$,

or $\frac{1}{\sqrt{a}} \cosh^{-1} \frac{\sqrt{a}\sqrt{R}}{\sqrt{(ac-b^2)}}$,

(29) $y = \sin^{-1} \sqrt{(\sin 2x)} \pm \sinh^{-1} \sqrt{(\sin 2x)}$,

$$\frac{dy}{dx} = \sqrt{(2 \tan x)}, \text{ or } \sqrt{(2 \cot x)}.$$

(30) $y = \tan^{-1} \sqrt{(\tanh x)} \pm \tanh^{-1} \sqrt{(\tanh x)}$,

$$\frac{dy}{dx} = \sqrt{(\coth x)}, \text{ or } \sqrt{(\tanh x)}.$$

(31) Prove that $\frac{dy}{dx} = \frac{1}{a+b \cos x}$, if

$$y = \frac{1}{\sqrt{(a^2-b^2)}} \cos^{-1} \frac{a \cos x + b}{a+b \cos x}$$

$$= \frac{2}{\sqrt{(a^2-b^2)}} \tan^{-1} \left\{ \left(\frac{a-b}{a+b} \right)^{\frac{1}{2}} \tan \frac{1}{2}x \right\}$$

or $\frac{1}{\sqrt{(b^2-a^2)}} \cosh^{-1} \frac{a \cos x + b}{a+b \cos x}$

$$= \frac{2}{\sqrt{(b^2-a^2)}} \tanh^{-1} \left\{ \left(\frac{b-a}{b+a} \right)^{\frac{1}{2}} \tan \frac{1}{2}x \right\}$$

$$= \frac{1}{\sqrt{(b^2-a^2)}} \log \frac{\sqrt{(b+a)} \cos \frac{1}{2}x + \sqrt{(b-a)} \sin \frac{1}{2}x}{\sqrt{(b+a)} \cos \frac{1}{2}x - \sqrt{(b-a)} \sin \frac{1}{2}x}.$$

(32) $\frac{dy}{dx} = \frac{1}{a+b \cosh x}$, if

$$y = \frac{1}{\sqrt{(b^2-a^2)}} \cos^{-1} \frac{a \cosh x + b}{a+b \cosh x}$$

$$= \frac{2}{\sqrt{(b^2-a^2)}} \tanh^{-1} \left\{ \left(\frac{b-a}{b+a} \right)^{\frac{1}{2}} \tanh \frac{1}{2}x \right\}$$

$$\begin{aligned}
 \text{or } & \frac{1}{\sqrt{(a^2-b^2)}} \cosh^{-1} \frac{a \cosh x + b}{a + b \cosh x} \\
 &= \frac{2}{\sqrt{(a^2-b^2)}} \tanh^{-1} \left\{ \left(\frac{a-b}{a+b} \right)^{\frac{1}{2}} \tanh \frac{1}{2} x \right\} \\
 &= \frac{1}{\sqrt{(a^2-b^2)}} \log \frac{\sqrt{(a+b) \cosh \frac{1}{2} x} + \sqrt{(a-b) \sinh \frac{1}{2} x}}{\sqrt{(a+b) \cosh \frac{1}{2} x} - \sqrt{(a-b) \sinh \frac{1}{2} x}}.
 \end{aligned}$$

$$(33) \quad y = \frac{1}{\sqrt{(a^2+b^2)}} \sinh^{-1} \frac{a \sinh x + b}{a + b \sinh x}, \quad \frac{dy}{dx} = \frac{1}{a + b \sinh x}.$$

(34) Denoting $a + b \cos x + c \sin x$ by P , prove that

$$\frac{dy}{dx} = \frac{1}{a + b \cos x + c \sin x} = \frac{1}{P},$$

$$\text{if } y = \frac{1}{\sqrt{(a^2-b^2-c^2)}} \cos^{-1} \frac{aP - a^2 + b^2 + c^2}{\sqrt{(b^2+c^2)P}}$$

$$\text{or } \frac{1}{\sqrt{(-a^2+b^2+c^2)}} \cosh^{-1} \frac{aP - a^2 + b^2 + c^2}{\sqrt{(b^2+c^2)P}}.$$

(35) Denoting $a + b \cosh x + c \sinh x$ by Q , prove that

$$\frac{dy}{dx} = \frac{1}{a + b \cosh x + c \sinh x} = \frac{1}{Q},$$

$$\text{if } y = \frac{1}{\sqrt{(-a^2+b^2-c^2)}} \cos^{-1} \frac{aQ - a^2 + b^2 - c^2}{\sqrt{(b^2-c^2)Q}},$$

$$\text{or } \frac{1}{\sqrt{(a^2-b^2+c^2)}} \cosh^{-1} \frac{aQ - a^2 + b^2 - c^2}{\sqrt{(b^2-c^2)Q}},$$

$$\text{or } \frac{1}{\sqrt{(a^2-b^2+c^2)}} \sinh^{-1} \frac{aQ - a^2 + b^2 - c^2}{\sqrt{(c^2-b^2)Q}}.$$

(36) Prove that $\frac{dy}{dx} = \frac{1}{(1+e \cos x)^2}$, if

$$(1-e^2)^{\frac{3}{2}}y = \sin^{-1} \frac{\sqrt{(1-e^2)} \sin x}{1+e \cos x} - \frac{e \sqrt{(1-e^2)} \sin x}{1+e \cos x},$$

$$\text{or } (e^2-1)^{\frac{3}{2}}y = -\sinh^{-1} \frac{\sqrt{(e^2-1)} \sin x}{1+e \cos x} + \frac{e \sqrt{(e^2-1)} \sin x}{1+e \cos x}.$$

$$(37) \quad y = \frac{1}{12} \log \frac{\sqrt{(4x^3-1)} + \sqrt{3}(2x-1)}{\sqrt{(4x^3-1)} + \sqrt{3}} \\ + \frac{1}{18} \sqrt{3} \tan^{-1} \frac{\sqrt{(4x^3-1)} - \sqrt{3}(x+1)}{3x},$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(4x^3-1)} + \sqrt{3}} \cdot \frac{x}{\sqrt{(4x^3-1)}}.$$

Write down the value of y when

$$\frac{dy}{dz} = \frac{1}{(z + \sqrt{3})^{\frac{2}{3}}(z^2+1)} \quad (\text{Euler}).$$

(38) Prove that the equation (Riccati's)

$$x \frac{dy}{dx} - ay + by^2 = cx^{2a}$$

is satisfied by

$$y = x^a \sqrt{\left(\frac{c}{b}\right) \tanh \left\{ \frac{x^a}{a} \sqrt{(-bc)} + C \right\}},$$

$$\text{or } x^a \sqrt{\left(-\frac{c}{b}\right) \tanh \left\{ \frac{x^a}{a} \sqrt{(-bc)} + C \right\}}.$$

(39) With $X = ax^4 + 4bx^3 + bcx^2 + 4b'x + a'$,

Y the same function of y , and

$$s = \frac{1}{4} \left(\frac{\sqrt{X} - \sqrt{Y}}{x-y} \right)^2 - \frac{1}{4} a(x+y)^2 - b(x+y) - c;$$

prove that, treating y as constant,

$$dx/\sqrt{X} = ds/\sqrt{(4s^3 - g_2s - g_3)},$$

where

$$g_2 = aa' - 4bb' + 3c^2, \quad g_3 = aca' + 2bcb' - ab'^2 - a'b^2 - c^3.$$

CHAPTER II.

INTEGRATION.

38. At this point it is advisable to introduce the idea of *Integration*, and to give a short preliminary sketch of its use in simple applications, reserving a more complete treatment for a subsequent chapter.

The process of *Integration* is the reverse of *Differentiation*, and is the province of the *Integral Calculus*. The Integral Calculus is required primarily for determining the area and length of a plane curve, the volume and surface of a solid, etc.

In the Differential Calculus a function y or fx is given, and we investigate rules for finding dy/dx or $f'x$.

But in the Integral Calculus the function dy/dx or $f'x$ is given, and we are required to find y or fx , the function of which $f'x$ is the differential co-efficient.

With the notation of the Differential Calculus

$$f'x = \frac{dfx}{dx} \dots\dots\dots (A),$$

or $f'x dx = dfx$, in the notation of *Differentials*.

Now supposing \int and d to represent inverse operations, so that \int and d cancel, and operating by \int on both sides,

$$\int f'x dx = \int dfx = fx \dots\dots\dots (B),$$

the notation of the Integral Calculus; the differential $f'x dx$ being called an *element* of the integral $\int f'x dx$ or fx .

Our object is to introduce the student to the integration symbol, the long \int , in conjunction with the symbol of differentiation d , at as early a stage as possible.

The process of Integration is of a tentative nature, depending on a previous knowledge of differentiation, as explained in Chapter I; just as Division in Arithmetic is a tentative process, depending on a knowledge of Multiplication and the Multiplication Table.

39. To every differentiation in the Differential Calculus there corresponds an integration in the Integral Calculus, and this correspondence for the functions we shall employ is exhibited in the following table.

DIFFERENTIAL CALCULUS.

$$(a) \frac{dx^n}{dx} = nx^{n-1}.$$

$$(b) \frac{d \sin mx}{dx} = m \cos mx.$$

$$(c) \frac{d \cos mx}{dx} = -m \sin mx.$$

$$(d) \frac{d \tan mx}{dx} = m \sec^2 mx.$$

$$(e) \frac{d \cot mx}{dx} = -m \operatorname{cosec}^2 mx.$$

$$(f) \frac{d \sec mx}{dx} = m \sec mx \tan mx.$$

$$(g) \frac{d \operatorname{cosec} mx}{dx} = -m \operatorname{cosec} mx \cot mx.$$

$$(h) \frac{d \operatorname{vers} mx}{dx} = m \sin mx.$$

$$(i) \frac{d \sin^{-1} x/a}{dx} = \frac{1}{\sqrt{(a^2 - x^2)}}.$$

$$(j) \frac{d \cos^{-1} x/a}{dx} = -\frac{1}{\sqrt{(a^2 - x^2)}}.$$

INTEGRAL CALCULUS.

$$\int x^m dx = \frac{x^{m+1}}{m+1},$$

except when $m = -1$, as in (v).

$$\int \cos mx dx = \frac{1}{m} \sin mx.$$

$$\int \sin mx dx = -\frac{1}{m} \cos mx.$$

$$\int \sec^2 mx dx = \frac{1}{m} \tan mx.$$

$$\int \operatorname{cosec}^2 mx dx = -\frac{1}{m} \cot mx.$$

$$\int \sec mx \tan mx dx = \frac{1}{m} \sec mx.$$

$$\int \operatorname{cosec} mx \cot mx dx = -\frac{1}{m} \operatorname{cosec} mx.$$

$$\int \sin mx dx = \frac{1}{m} \operatorname{vers} mx.$$

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1} x/a.$$

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = -\cos^{-1} x/a.$$

$$(k) \frac{d \tan^{-1} x/a}{dx} = \frac{a}{a^2 + x^2}.$$

$$(l) \frac{d \cot^{-1} x/a}{dx} = -\frac{a}{x^2 + a^2}.$$

$$(m) \frac{d \sec^{-1} x/a}{dx} = \frac{a}{x\sqrt{(x^2 - a^2)}}.$$

$$(n) \frac{d \operatorname{cosec}^{-1} x/a}{dx} = -\frac{a}{x\sqrt{(x^2 - a^2)}}.$$

$$(o) \frac{d \operatorname{vers}^{-1} x/a}{dx} = \frac{1}{\sqrt{(2ax - x^2)}}.$$

$$(p) \frac{d a^x}{dx} = a^x \log a.$$

$$(q) \frac{d e^{cx}}{dx} = c e^{cx}.$$

$$(r) \frac{d \sinh mx}{dx} = m \cosh mx.$$

$$(s) \frac{d \cosh mx}{dx} = m \sinh mx.$$

$$(t) \frac{d \tanh mx}{dx} = m \operatorname{sech}^2 mx.$$

$$(u) \frac{d \coth mx}{dx} = -m \operatorname{cosech}^2 mx.$$

$$(v) \frac{d \log x}{dx} = \frac{1}{x}.$$

$$(w) \frac{d \sinh^{-1} x/a}{dx} = \frac{1}{\sqrt{(a^2 + x^2)}}.$$

$$(x) \frac{d \cosh^{-1} x/a}{dx} = \frac{1}{\sqrt{(x^2 - a^2)}}.$$

$$(y) \frac{d \tanh^{-1} x/a}{dx} = \frac{a}{a^2 - x^2}.$$

$$(z) \frac{d \coth^{-1} x/a}{dx} = -\frac{a}{x^2 - a^2}.$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} x/a.$$

$$\int \frac{dx}{x^2 + a^2} = -\frac{1}{a} \cot^{-1} x/a.$$

$$\int \frac{dx}{x\sqrt{(x^2 - a^2)}} = \frac{1}{a} \sec^{-1} x/a.$$

$$\int \frac{dx}{x\sqrt{(x^2 - a^2)}} = -\frac{1}{a} \operatorname{cosec}^{-1} x/a.$$

$$\int \frac{dx}{\sqrt{(2ax - x^2)}} = \operatorname{vers}^{-1} x/a.$$

$$\int a^x dx = \frac{a^x}{\log a}.$$

$$\int e^{cx} dx = \frac{1}{c} e^{cx}.$$

$$\int \cosh mx dx = \frac{1}{m} \sinh mx.$$

$$\int \sinh mx dx = \frac{1}{m} \cosh mx.$$

$$\int \operatorname{sech}^2 mx dx = \frac{1}{m} \tanh mx.$$

$$\int \operatorname{cosech}^2 mx dx = -\frac{1}{m} \coth mx.$$

$$\int \frac{dx}{x} = \log x, \text{ or } \log(-x).$$

$$\int \frac{dx}{\sqrt{(a^2 + x^2)}} = \sinh^{-1} x/a, \\ \text{or } \log\{\sqrt{(a^2 + x^2)} + x\}.$$

$$\int \frac{dx}{\sqrt{(x^2 - a^2)}} = \cosh^{-1} x/a, \\ \text{or } \log\{x + \sqrt{(x^2 - a^2)}\}.$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} x/a \quad (x^2 < a^2), \\ \text{or } \frac{1}{2a} \log \frac{a+x}{a-x}, \text{ or } \frac{1}{2a} \log \frac{x+a}{x-a}.$$

$$\int \frac{dx}{x^2 - a^2} = -\frac{1}{a} \coth^{-1} x/a \quad (x^2 > a^2), \\ \text{or } \frac{1}{2a} \log \frac{x-a}{x+a}, \text{ or } \frac{1}{2a} \log \frac{a-x}{a+x}.$$

Since the d.c. of a constant is zero, therefore in integration an arbitrary constant may be added, and the integral is then called an *indefinite* integral.

When in the above formulas a discrepancy of results in integration appears, as between (c) and (h), (i) and (j), (k) and (l), (m) and (n), and in (v), (w), (x), (y), (z), the results will be found to differ by a constant.

Again, since the d.c. of a sum of functions is the sum of their d.c.'s, it follows that the integral of a sum is the sum of the integrals of the functions, and so on; so that

$$\int (au + bv + \dots) dx = a \int u dx + b \int v dx + \dots$$

40. Integration by substitution.

Any function of x which can be thrown into the form $(fx)^m f'x$, where $f'x$ is the derivative of fx , is immediately integrable by (a), because

$$\int (fx)^m f'x dx = \frac{(fx)^{m+1}}{m+1}.$$

In such cases the result is sometimes more immediately obvious if we substitute for fx a new variable y ; and since $f'x dx = dy$, in the notation of differentials, then,

$$\int (fx)^m f'x dx = \int y^m dy = \frac{y^{m+1}}{m+1} = \frac{(fx)^{m+1}}{m+1},$$

as before.

Such an operation is called *integration by substitution*; and to discover the most appropriate substitution is one of the artifices of the Integral Calculus.

In the exceptional case of $m = -1$, then by (v),

$$\int \frac{f'x}{fx} dx = \int \frac{dy}{y} = \log y = \log fx;$$

thus
$$\int \frac{2x+1}{x^2+x+1} dx = \log(x^2+x+1).$$

Again, by (k),

$$\int \frac{f' dx}{a^2 + (fx)^2} = \int \frac{dy}{a^2 + y^2} = \frac{1}{a} \tan^{-1} \frac{y}{a} = \frac{1}{a} \tan^{-1} \frac{fx}{a};$$

thus
$$\int \frac{dx}{x^2 + x + 1} = \int \frac{dx}{(x + \frac{1}{2})^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{x + \frac{1}{2}}{\frac{1}{2}\sqrt{3}}.$$

Examples.—Integrate with respect to x ,

(1) $0, 1, a, x, ax + b, x^2, ax^2 + 2bx + c, x^3, x^4, \dots, x^n, (nx)^{\frac{1}{n}-1},$

$$\sqrt{x}, \sqrt[3]{x}, \frac{1}{\sqrt{x}}, \frac{1}{\sqrt[3]{x}}, \frac{1}{x}, \frac{1}{x^{\frac{3}{2}}}, \frac{1}{x^{\frac{4}{3}}}, \frac{1}{x^2}, \frac{1}{x^n}, \frac{1}{(nx)^{(n-1)/n}},$$

$$A + Bx + Cx^2 + Dx^3 + \dots, ax^m + bx^n + cx^p + \dots$$

(2) $x + a, (x + a)^2, (x + a)^n, \sqrt{x + a}, \frac{1}{\sqrt{x + a}}, \frac{1}{x + a},$

$$\frac{1}{(x + a)^{\frac{3}{2}}}, (mx + n)^p, \frac{A}{x - a}, \frac{B}{(x - b)^p}.$$

(3) $x^2 + a^2, (x^2 + a^2)^3, x(x^2 - a^2)^n, x\sqrt{x^2 + a^2}, \frac{x}{\sqrt{x^2 + a^2}},$

$$\frac{x}{x^2 - a^2}, \frac{1}{x^2 + a^2}, \frac{1}{x^2 - a^2}, \frac{x}{(x^2 + a^2)^{\frac{3}{2}}}.$$

(4) $\frac{2x + 1}{x^2 + x + 1}, \frac{1}{x^2 + x + 1}, \frac{x + 2}{x^2 + x + 1}, \frac{Px + Q}{x^2 + x + 1}, \frac{x + 2}{x^2 + 2x + 2},$

$$\frac{x - \sqrt{2}}{x^2 - \sqrt{2}x + 1}, \frac{2}{4x^2 + 3}, \frac{1}{ax^2 + c}, \frac{Px + Q}{ax^2 + c}.$$

(5) $\frac{x - c}{\sqrt{\{(x - c)^2 - a^2\}}}, \frac{x - c}{(x - c)^2 + a^2}, \frac{1}{(x - c)^2 + a^2}, \frac{Px + Q}{(x - c)^2 + a^2},$
 $\{(x - c)^2 + a^2\}^n(x - c).$

(6) $\frac{ax + b}{\sqrt{ax^2 + 2bx + c}}, \frac{ax + b}{ax^2 + 2bx + c}, (ax^2 + 2bx + c)^m(ax + b).$

- (7) $\frac{1}{\sqrt{(x+a)} + \sqrt{(x+b)}}$, (rationalize the denominator),
 $\frac{1}{x^4 \sqrt{(x^2+1)}}$ (substitute $x^2 = \frac{1}{y}$), $\frac{1}{x \sqrt{(ax^n+b)}}$
 (substitute $\sqrt{(ax^n+b)} = y$).
- (8) $\frac{(\log x)^n}{x}$, $\frac{1}{x \log x}$, $\frac{x^{n-1}}{ax^n+b}$, $\frac{\sin x}{a+b \cos x}$, $\frac{f'x}{a+bf'x}$, $\frac{f'x}{(a+bf'x)^p}$,
 $(ax^n+b)^{-1-1/n}$ (substitute $ax^n+b = x^ny^n$).
- (9) $\cos^2 x$, $\cos^3 x$, $\cos^4 x$, ...; $\sin^2 x$, $\sin^3 x$, $\sin^4 x$...; $\cos mx$
 $\cos nx$, $\sin mx \sin nx$, $\cos mx \sin nx$.

(Convert the powers and products of $\cos x$ and $\sin x$ into cosines and sines of multiples of x ; thus

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x), \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x);$$

$$\cos^3 x = \frac{1}{4} \cos 3x + \frac{3}{4} \cos x, \quad \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x,$$

$$\cos^4 x = \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x,$$

$$\sin^4 x = \frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x;$$

$$\cos mx \cos nx = \frac{1}{2} \cos(m-n)x + \frac{1}{2} \cos(m+n)x;$$

$$\sin mx \sin nx = \frac{1}{2} \cos(m-n)x - \frac{1}{2} \cos(m+n)x;$$

$$\cos mx \sin nx = \frac{1}{2} \sin(m+n)x - \frac{1}{2} \sin(m-n)x;$$

and now the integrals with respect to x can be immediately written down).

The answers of these examples are not given, because it is better practice in integration for the student to discover the results for himself; the correctness of a result can always be tested by differentiation.

41. Quadrature.

The Integral Calculus was invented for the purpose of finding areas, or for *quadrature*, as it was formerly called; and the meaning of Integration is best illustrated by its application to finding the area of a curve.

Let $y = fx$ (fig. 17) be the equation of a curve CPD ; then if $OM = x$, $MP = fx$.

Let the area $AMPC$ between the curve and the axis of x , bounded by an initial ordinate AC and the variable ordinate MP , be denoted by A ; keeping AC fixed and varying MP , A is some function of x , which it is required to determine.

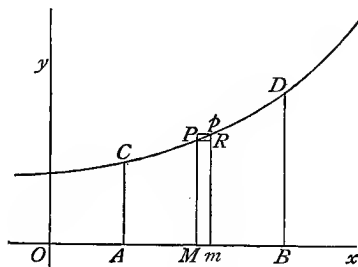


Fig. 17

Let the ordinate MP be moved into the adjacent position mp , by giving x the increment Δx ; and let ΔA be the corresponding increment of A , and Δy of y .

Then $Mm = \Delta x$, $mp = y + \Delta y$; and the area $MmpP = \Delta A$.

But $MmpP$ lies between the rectangles Pm and Mp , and therefore ΔA lies between $y\Delta x$ and $(y + \Delta y)\Delta x$; or, $\Delta A/\Delta x$ lies between y and $y + \Delta y$.

Proceeding to the limit by making Δx , and therefore Δy and ΔA indefinitely small,

$$\frac{dA}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta A}{\Delta x} = y = fx.$$

Therefore

$$\begin{aligned} A &= \int y dx + \text{a constant} \\ &= \int fx dx + \text{a constant}; \end{aligned}$$

so that to determine A we must know how to integrate fx .

42. *Fluxions and Fluents.*

According to the *Doctrine of Fluxions*, the old-fashioned name of the Calculus in this country, the area $AMPC$, called the *fluent*, is supposed to be generated or to grow by the flow or motion of the variable ordinate MP ; and the rate of growth of the area, called its *fluxion*, is equal to the ordinate MP into the fluxion of x ; with Leibnitz's notation,

$$\frac{dA}{dt} = y \frac{dx}{dt}, \text{ or } \frac{dA}{dx} = y,$$

so that

$$A = \int y dx.$$

Generally, "Mathematical quantity, particularly extension, may be conceived as generated by continued local motion (as in the growth of a tree); and all quantities whatever, at least by analogy and accommodation, may be conceived as generated after a like manner."

(Sir Isaac Newton, *Fluxions*, edited by Colson, 1736.)

"The Differential Method of Leibnitz teaches us to consider Magnitudes as made up of an infinite Number of very small constituent parts, put together; whereas the Fluxionary Method of Newton teaches us to consider Magnitudes as generated by Motion.

A Line is described, and in describing is generated, not by an apposition of Points, or Differentials, but by the Motion or Flux of a Point."

(*Doctrine of Fluxions*, by J. Hodgson, 1736.)

Thus if the point M is supposed to move in the direction MP , it will trace out the line MP .

If the variable ordinate MP moves parallel to itself, it will trace out the area $ABDC$.

Similarly, any solid may be supposed generated by the motion of a variable plane area perpendicularly to itself; and then $dV/dt = A dx/dt$, if V denotes the volume, A the variable plane area, and dx/dt the velocity of the plane perpendicular to itself.

For instance, if the volume V is bounded by the surface formed by the revolution of the curve $y = fx$ round the axis of x , then the fluent V may be supposed generated by the motion of an expanding (or contracting) circle of radius y , as in boring a hole of varying diameter, or turning a body in a lathe; and therefore

$$dV/dx = \pi y^2, \text{ or } V = \pi \int y^2 dx.$$

43. *Corrected Integrals.*

Denoting the indefinite integral $\int f_x dx$ by $f_1 x$, then

$$A = f_1 x + C,$$

where C denotes a constant; and denoting OA by a , then since $A = 0$ when $x = a$, therefore $C = -f_1 a$, and

$$A = f_1 x - f_1 a.$$

This is expressed by the notation

$$A = \int_a^x f_x dx, \text{ or simply } \int_a^x f_x dx = (f_1 x)_a^x = f_1 x - f_1 a,$$

a being called the *lower limit*, and the integral is then called a *corrected* integral.

Sometimes the fixed ordinate is taken to the right of the variable ordinate MP , at BD suppose, where $OB = b$; and then the area $MBDP$ is given by

$$\int_x^b f_x dx, \text{ or simply } \int_x^b f_x dx = (f_1 x)_x^b = f_1 b - f_1 x,$$

and b is called the *upper limit*.

Thus, the integrals of § 39, when corrected, become

$$\int_a^x x^m dx = \left(\frac{x^{m+1}}{m+1} \right)_a^x = \frac{x^{m+1} - a^{m+1}}{m+1},$$

$$\int_x^b x^m dx = \left(\frac{x^{m+1}}{m+1} \right)_x^b = \frac{b^{m+1} - x^{m+1}}{m+1}.$$

$$\int x^{-m} dx = \frac{1}{(m-1)x^{m-1}}.$$

$$\int_0^x \cos x dx = (\sin x)_0^x = \sin x.$$

$$\int_0^x \sin x dx = (-\cos x)_0^x = 1 - \cos x = \text{vers } x.$$

$$\int \sin x dx = -\cos x.$$

$$\int \cos x dx = \sin x.$$

$$\int \sec^2 x dx = \tan x.$$

$$\int \csc^2 x dx = -\cot x.$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}.$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \cos^{-1} \frac{x}{a}.$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}.$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \cot^{-1} \frac{x}{a}.$$

$$\int a^x dx = \frac{a^x - 1}{\log a}.$$

$$\int \frac{dx}{x} = \log \frac{x}{a}.$$

$$\int \cosh u du = \sinh u. \quad \int \sinh u du = \cosh u - 1 = \text{versh } u, \text{ etc.}$$

44. Another geometrical interpretation of integration is given here, adapted from Newton's Lemma II., *Principia*, Lib. I., § 1.

Let the area $ABDC$ (fig. 18) bounded by the curve $y = fx$, the axis of x , and the initial and final ordinates

AC and BD , be divided into a large number of narrow strips like $PMmp$ by equidistant ordinates at distance Δx .

Then the difference between the external rectangle Mp and the internal rectangle Pm is the rectangle Pp or Qq ; and therefore the difference between the sum of all the external rectangles and the sum of all the internal rectangles so described is the rectangle DE ; also the area $ABDC$ is intermediate to the sum of the external and the sum of the internal rectangles.

Now in the limit when the breadth Δx of the rectangles is indefinitely diminished and their number proportionately increased, the rectangle DE vanishes, and

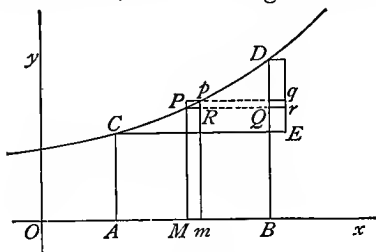


Fig.18

therefore the sum of all the external and the sum of all the internal rectangles each become equal to the area $ABDC$.

If $OM = x$, $MP = y$, then the rectangle $Pm = y\Delta x$; and denoting the sum of all the internal rectangles by $\Sigma y\Delta x$,

$$\begin{aligned} \text{then the area } ABDC &= \lim_{\Delta x \rightarrow 0} \sum_{x=a}^{x=b} y\Delta x, \\ &= \int_a^b y dx = \int_a^b f(x) dx, \end{aligned}$$

replacing Σ by \int , and Δx by dx in the limit, and supposing $OA = a$, $OB = b$.

With oblique coordinate axes, inclined at the angle ω ,

$$\text{the area } ABDC = \sin \omega \int_a^b y dx.$$

Any small part, such as ΔA or $y\Delta x$, is called a *finite element* of area, the symbol Δ (Delta) denoting as in § 8 a small finite increment; and the symbol Σ (Sigma) is used, in conjunction with Δ , to denote the sum of a number of finite elements, such as $\Sigma y\Delta x$, representing approximately the area $ABDC$; and the transition to the Integral Calculus, where approximation ceases and the area is given exactly, is represented by replacing Σ by \int and Δ by d .

In practical problems of engineering and shipbuilding where we have recourse to the method of *approximate quadrature*, we divide the area $ABDC$ into a finite number of elements $MmpP$, of which only one side Pp is curved, and then Pp may be made sufficiently small for it to be taken as straight without sensible error; and now the area $MmpP$ may be taken as the arithmetic mean of the outer rectangle Mp and the inner rectangle Pm , and the whole area $ABDC$ as the arithmetic mean of the sum of the outer and of the inner rectangles.

The student is recommended (by De Morgan) never to lose sight of the manner in which he would perform an integration, represented, graphically, by a quadrature, if a rough solution for practical purposes only was required.

Rules for approximate quadrature, Simpson's, Weddle's, etc., will be given hereafter; but meanwhile the arithmetic mean of a number of equidistant ordinates may be taken as the mean ordinate, which gives the height of the approximately equivalent rectangle.

45. *Integration between Limits. Definite Integrals.*

Denoting the indefinite integral $\int f x d x$ by $f_1 x$, then

$$\int_a^b f x d x = (f_1 x)_a^b = f_1 b - f_1 a,$$

and this is now called a *definite integral*; a being called the *lower limit* and b the *upper limit*.

(The term *definite integral* is however retained *par excellence* for integrals which can only be evaluated between certain definite limits, and of which the indefinite integrals cannot be found.)

Examples.—Prove that the definite integrals

$$(1) \int_a^b x^n d x = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

$$(2) \int_0^{\frac{1}{2}\pi} \cos \theta d \theta = \int_0^{\frac{1}{2}\pi} \sin \theta d \theta = 1.$$

$$(3) \int_0^{\frac{1}{2}\pi} \cos^2 \theta d \theta = \int_0^{\frac{1}{2}\pi} \sin^2 \theta d \theta = \frac{1}{4} \pi.$$

$$(4) \int_0^{\frac{1}{2}\pi} (\cos \theta)^3 d \theta = \int_0^{\frac{1}{2}\pi} (\sin \theta)^3 d \theta = \frac{2}{3}.$$

$$(5) \int_0^{\frac{1}{2}\pi} (\cos \theta)^4 d \theta = \int_0^{\frac{1}{2}\pi} (\sin \theta)^4 d \theta = \frac{3}{16} \pi.$$

$$(6) \int_0^1 \frac{d x}{1+x^2} = \int_1^{\infty} \frac{d x}{x^2+1} = \frac{1}{4} \pi.$$

$$(7) \int_0^1 \frac{d x}{3+x^2} = \int_1^3 \frac{d x}{3+x^2} = \int_3^{\infty} \frac{d x}{3+x^2} = \frac{1}{18} \sqrt{3} \pi.$$

$$(8) \int_0^a \frac{d x}{a^2+x^2} = \int_a^{\infty} \frac{d x}{x^2+a^2} = \frac{\pi}{4a}.$$

$$(9) \int_0^{\infty} \frac{x^2 d x}{(x^2+a^2)(x^2+b^2)} = \frac{\frac{1}{2} \pi}{a+b}.$$

46. An interchange of limits changes the sign of a definite integral; for

$$\int_b^a f x dx = f_1 a - f_1 b = - \int_a^b f x dx.$$

Again, when we evaluate the integral by means of a substitution, say $x = F\phi$, we must be careful to change the limits at the same time to the corresponding values of the new variable ϕ .

Thus if $\phi = \alpha$ makes $x = a$, and $\phi = \beta$ makes $x = b$, then

$$\int_a^b f x dx = \int_{\alpha}^{\beta} f x \frac{dx}{d\phi} d\phi = \int_{\alpha}^{\beta} f(F\phi) F' \phi d\phi,$$

equivalent to $f_1 b - f_1 a = f_1(F\beta) - f_1(F\alpha)$.

For instance, if we evaluate the definite integral

$$\int_0^a \sqrt{(a^2 - x^2)} dx \text{ by means of the substitution } x = a \sin \phi,$$

then $\phi = 0$ makes $x = 0$, and $\phi = \frac{1}{2}\pi$ makes $x = a$, while $dx = a \cos \phi d\phi$; so that

$$\int_0^a \sqrt{(a^2 - x^2)} dx = \int_0^{\frac{1}{2}\pi} a^2 \cos^2 \phi d\phi = \frac{1}{2} a^2 \int_0^{\frac{1}{2}\pi} (1 + \cos 2\phi) d\phi = \frac{1}{4} \pi a^2.$$

Considerations of *symmetry* and *periodicity* of the function $f x$ to be integrated are useful.

Thus if $f x$ is an *even* function of x , so that $f(-x) = f x$,

then
$$\int_{-a}^a f x dx = 2 \int_0^a f x dx;$$

but if $f x$ is an *odd* function, so that $f(-x) = -f x$, then

$$\int_{-a}^a f x dx = 0.$$

Thus $x^2, x^4, x^{2n}, \cos x, \sec x, \text{vers } x$ are even functions of x ; while $x, x^3, x^{2n+1}, \sin x, \tan x, \cot x, \text{cosec } x$ are odd functions.

For instance,
$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\cos x)^{2n+1} dx = 2 \int_0^{\frac{1}{2}\pi} (\cos x)^{2n+1} dx,$$

but
$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\sin x)^{2n+1} dx = 0;$$

and
$$\int_0^{\pi} (\sin x)^{2n+1} dx = 2 \int_0^{\frac{1}{2}\pi} (\sin x)^{2n+1} dx,$$

but
$$\int_0^{\pi} (\cos x)^{2n+1} dx = 0.$$

Again, if fx is a *periodic function of period l* , so that $f(x+nl) = fx$, where n is an integer, (for instance, $\sin 2\pi x/l$ or $\cos 2\pi x/l$), then

$$\int_0^{nl} fx dx = n \int_0^l fx dx; \text{ and } \int_0^{n+l} fx dx = n \int_0^l fx dx + \int_0^a fx dx.$$

Thus the period of $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$, $\operatorname{vers} x$, or any odd power of these functions is 2π ; the period of any even power, or of $\tan x$, $\cot x$, or an odd power of $\tan x$ or $\cot x$, is π ; the period of any even power of $\tan x$ or $\cot x$ is $\frac{1}{2}\pi$.

It is advisable, in integration, to make use of these considerations in order to keep the limits of integration as close together as possible.

Examples.—

(1) Evaluate $\int_{-a}^a (1, x, x^2, x^3, \dots, x^{2n}, x^{2n+1}) dx$.

(2) $\int (\cos \theta)^n d\theta$ and $\int (\sin \theta)^n d\theta$, for $n = 1, 2, 3, 4, \dots$,
between the limits $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, 0 and π , 0 and $\frac{3}{2}\pi$, 0 and 2π .

(3) Prove that

(i.) $\int_0^a x \sqrt{a-x} dx = \frac{4}{15} a^{\frac{5}{2}}$. (ii.) $\int_0^a x \sqrt{a^2-x^2} dx = \frac{1}{3} a^3$.

$$(iii.) \int_0^a \frac{dx}{\sqrt{(a^2 - x^2)}} = \frac{1}{2}\pi. \quad (iv.) \int_0^a \frac{x^2 dx}{\sqrt{(a^2 - x^2)}} = \frac{1}{4}\pi a^2.$$

$$(v.) \int_0^a \frac{dx}{\sqrt{(ax - x^2)}} = \pi \text{ (substitute } x = a \sin^2 \theta \text{).}$$

$$(vi.) \int_0^a \sqrt{\left(\frac{a-x}{x}\right)} dx = \frac{1}{2}\pi a. \quad (vii.) \int_0^a \sqrt{(ax - x^2)} dx = \frac{1}{8}\pi a^2.$$

$$(viii.) \int_0^a x \sqrt{(ax - x^2)} dx = \frac{1}{16}\pi a^3.$$

(4) Prove geometrically that

$$\int_0^a f x dx = \int_0^a f(a-x) dx, \quad \int_a^b f x dx = \int_a^b f(a+b-x) dx.$$

47. Centre of Gravity or Centroid of an Area.

The centre of gravity, or, as it is now called, the *centroid* G of the area OMP , where OP is any curve (fig. 19), is determined by integration as follows: denoting the coordinates KG , HG of the centroid G by \bar{x} , \bar{y} , then the moment about the axes of the whole area A collected at G is equal to the sum of the moments of the separate elements, such as mp' , which may be supposed to have its centroid ultimately at g , the middle point of mp .

$$\text{Therefore} \quad \bar{x}A = \int_0^a xy dx, \quad \bar{y}A = \int_0^a \frac{1}{2}y^2 dx;$$

taking x and $\frac{1}{2}y$ as the coordinates of the centroid of the element $y dx$; whence the values of \bar{x} and \bar{y} are found.

It must be noticed in these integrals, as in all corrected integrals, that under the sign of integration x , y must be supposed to denote the coordinates Om , mp of any intermediate point p of the curve OP ; but that when the integration is performed, then x , y represent OM , MP the coordinates of P , the variable upper limit of

the integration; this will not however be found to create confusion.

Generally, for the centroid G of the area $ABDC$ (fig. 18)

$$\bar{x}A = \int_a^b xy dx, \quad \bar{y}A = \int_a^b \frac{1}{2}y^2 dx.$$

48. *Application of the Integral Calculus to the quadrature of the parabola $y^2 = px$ (fig. 19).*

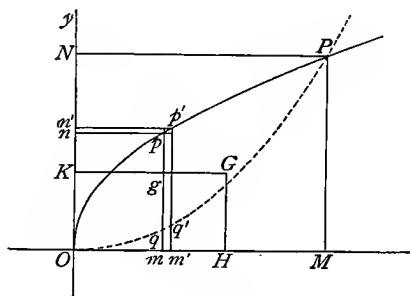


Fig. 19

$$\begin{aligned} \text{The area } OMP &= \int_0^x y dx = p^{\frac{1}{2}} \int_0^x x^{\frac{1}{2}} dx \\ &= \frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}} = \frac{2}{3} xy = \frac{2}{3} \text{ rectangle } OMPN. \end{aligned}$$

Similarly,

$$\begin{aligned} \text{the area } ONP &= \int_0^y x dy = \int_0^y \frac{y^2 dy}{p} = \frac{1}{3} \frac{y^3}{p} = \frac{1}{3} xy \\ &= \frac{1}{3} \text{ rectangle } OMPN. \end{aligned}$$

For the parabolic area OMP , $A = \frac{2}{3} xy = \frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}}$;

$$\bar{x}A = \int_0^x p^{\frac{1}{2}} x^{\frac{3}{2}} dx = \frac{2}{5} p^{\frac{1}{2}} x^{\frac{5}{2}}, \quad \bar{x} = \frac{3}{5} x = \frac{3}{5} OM;$$

$$\bar{y}A = \int_0^y \frac{1}{2} px dx = \frac{1}{4} px^2, \quad \bar{y} = \frac{3}{8} p^{\frac{1}{2}} x^{\frac{1}{2}} = \frac{3}{8} MP.$$

For the centroid of the parabolic area ONP , taking y as the independent variable, the area $B = \frac{1}{3}xy = \frac{1}{3}y^3/p$;

$$\bar{x}B = \int_0^p \frac{1}{2}x^2 dy = \int_0^p \frac{1}{2}y^4 dy / p^2 = \frac{1}{10}y^5 / p^2, \quad \bar{x} = \frac{3}{10}y^2 / p = \frac{3}{10}x;$$

$$\bar{y}B = \int_0^p xy dx = \int_0^p y^3 dy / p = \frac{1}{4}y^4 p, \quad \bar{y} = \frac{3}{4}y.$$

Similarly, it may be proved that if the equation of the curve OP (fig. 19) is $y^n = p^n - mx^m$, the area OMP is

$$\frac{nxy}{m+n}, \text{ and the coördinates of its centroid are } \frac{m+n}{m+2n}x,$$

$$\frac{m+n}{4m+2n}y; \text{ and that the area } ONP \text{ is } \frac{mxy}{m+n}, \text{ and the co-}$$

$$\text{ordinates of its centroid are } \frac{m+n}{2m+4n}x, \frac{m+n}{2m+n}y.$$

For instance, the whole area cut off from the curve $9xy^2 = 4x^3$ (fig. 2 i.) by the line $x=a$ is $\frac{8}{15}a^2$, and $\bar{x} = \frac{5}{7}a$.

49. In finding the area between two curves, the graphs of $y_1 = f_1x$ and $y_2 = f_2x$, cut off by two ordinates given by $x=a$ and $x=b$, we divide the area into elementary strips by lines parallel to Oy ; so that an element of area is ultimately $(y_1 - y_2)dx$, and the coordinates of the centroid of the element are x and $\frac{1}{2}(y_1 + y_2)$; and thus the area

$$A = \int_a^b (y_1 - y_2)dx,$$

$$\text{and } \bar{x}A = \int_a^b x(y_1 - y_2)dx, \quad \bar{y}A = \int_a^b \frac{1}{2}(y_1^2 - y_2^2)dx,$$

giving \bar{x} , \bar{y} , the coordinates of the centroid of the area.

Similarly in finding the area between the two curves cut off by two lines $y=a$ and $y=\beta$, we divide the area into elements $(x_2 - x_1)dy$, having the centroid at $\frac{1}{2}(x_2 + x_1)$, y ; and now the area

$$B = \int_a^{\beta} (x_2 - x_1) dy,$$

$$\text{and } \bar{x}B = \int \frac{1}{2}(x_2^2 - x_1^2) dy, \quad \bar{y}B = \int y(x_2 - x_1) dy;$$

as is manifest on drawing a figure, such as fig. 19.

When the whole area between two curves is required, the limits of integration are determined by the co-ordinates of the points of intersection of the curves.

Examples.

(1) Prove that the area between the parabolas $y^2 = px$ and $x^2 = qy$ is $\frac{1}{3}pq$, and the coordinates of its centroid are $\bar{x} = \frac{9}{20}p^{\frac{2}{3}}q^{\frac{1}{3}}$, $\bar{y} = \frac{9}{20}p^{\frac{1}{3}}q^{\frac{2}{3}}$.

(2) Prove that the area between $y^m = x^n$ and $y^n = x^m$ is $(m-n)/(m+n)$, and that the coordinates of its centroid are $\bar{x} = \bar{y} = \frac{(m+n)^2}{(m+2n)(2m+n)}$.

$$\text{Also for the curves } \left(\frac{y}{b}\right)^m = \left(\frac{x}{a}\right)^n, \quad \left(\frac{y}{b}\right)^p = \left(\frac{x}{a}\right)^q.$$

(3) Determine the work done by the expansion of steam on a piston in a cylinder, or by the expansion of powder gas on a shot in the bore of a gun, supposing the pressure to vary inversely as the m th power of the volume.

(Let P denote the initial pressure, in lb. per sq. inch, of the steam or powder gas when occupying a length a feet of the cylinder or bore; then when the piston or shot has advanced a distance $x-a$ feet, the pressure will be $P(a/x)^m$, and the work done will be, in foot-pounds,

$$\int_a^x \frac{1}{4}\pi d^2 P(a/x)^m dx = \frac{1}{4}\pi d^2 P a \frac{1 - (a/x)^{m-1}}{m-1},$$

if the diameter of the cylinder or bore is d inches.

Representing the pressure graphically by the ordinates of the curve CPD in fig. 20, then the area $AMPC$ will represent the work done per square inch of cross section of the cylinder or bore.

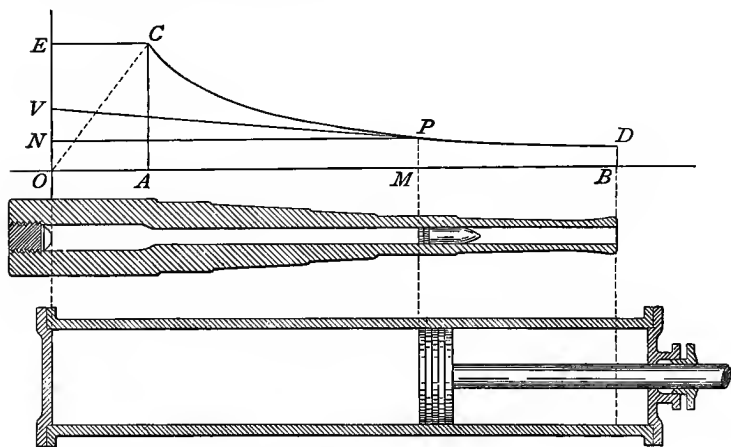


Fig.20

The *mean effective pressure* is obtained by dividing the work done by the distance of advance, so that the quadrature of the area $ABDC$ leads to the value of the mean effective pressure; and where the curve CPD is given by an Indicator, the method of approximate quadrature of § 44, or else a Planimeter is employed.

When $m=1$, the pressure varies inversely as the volume, as in Boyle's law, and the above expression for the work done becomes illusory or indeterminate.

But, by expanding $(a/x)^{m-1}$ in powers of $m-1$ by § 31,

$$Pa \frac{1 - (a/x)^{m-1}}{m-1} = Pa \left\{ \log \frac{x}{a} - \frac{m-1}{2} \left(\log \frac{x}{a} \right)^2 + \dots \right\},$$

reducing to $Pa \log x/a$, when $m=1$.

But, independently, in this case the work done is

$$\int_a^x \frac{Pa}{x} dx = Pa \log \frac{x}{a};$$

while the pressure curve CPD becomes a hyperbola; and thus the area $AMPC$ is to the rectangle $OACE$ as $\log OM/OA$ to unity.

Natural logarithms were formerly called *hyperbolic* logarithms for this reason, because they "square the hyperbolic areas"; but the fact that all other systems of logarithms are equally connected with the hyperbola was not perceived when this name was given.

The cross section of a gun has been drawn (fig. 20), taking OA as the length of the cartridge; but in the cylinder of a steam engine, the piston M starts from a point close to O , and A must be taken to represent the point of cut-off of the steam, the pressure from O to A being taken as the full boiler pressure P .

The work done on the piston per square inch of cross section, in going from O to M will now be represented by the area $OMPCE = Pa(1 + \log x/a) = Pa \log ex/a$; and if we cut off the triangles OMP , OCE , each equal to $\frac{1}{2}Pa$, we are left with the sectorial area $OPC = Pa \log x/a = \text{area } AMPC$, an important property of the hyperbola).

(4) In the curve $y = \sin x$, the area $OMP = \text{vers } x$, and in the curve $y/b = \sin x/a$, the area $OMP = ab \text{ vers } x/a$.

(5) Prove that in the exponential curve $y = a^x$ (fig. 11) the area between the curve and the axis of x , cut off to the left by the ordinate MP

$$= a^x / \log a = \text{rectangle } PM, MT;$$

and in the catenary (fig. 15) $y/a = \cosh x/a$, the area $OMPA = a^2 \sinh x/a$.

50. *The Quadrature of the Circle and Ellipse.*

In the circle (fig. 21),

$$x^2 + y^2 = a^2, \text{ or } y = \sqrt{(a^2 - x^2)};$$

and therefore the area

$$OMPB = \int_0^x \sqrt{(a^2 - x^2)} dx \dots\dots\dots (i.);$$

and the area $PMA = \int_x^a \sqrt{(a^2 - x^2)} dx \dots\dots\dots (ii.).$

To find the first integral (i.), substitute

$$x = a \sin \phi, \text{ then the angle } BOP = \phi;$$

and $y = \sqrt{(a^2 - x^2)} = a \cos \phi, dx = a \cos \phi d\phi.$

Therefore the area $OMPB$

$$\begin{aligned} &= \int_0^\phi a^2 \cos^2 \phi d\phi = \frac{1}{2} a^2 \int_0^\phi (1 + \cos 2\phi) d\phi \\ &= \frac{1}{2} a^2 \left(\phi + \frac{1}{2} \sin 2\phi \right) = \frac{1}{2} a^2 \phi + \frac{1}{2} a^2 \sin \phi \cos \phi; \end{aligned}$$

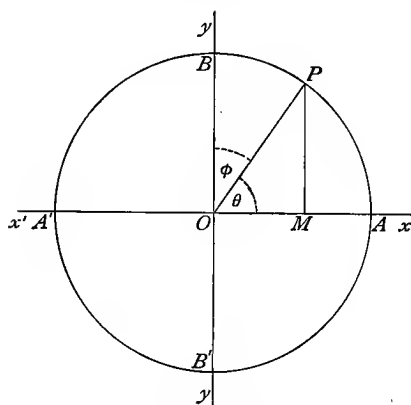


Fig. 21

of which the sector $OPB = \frac{1}{2} a^2 \phi$, and the triangle $OMP = \frac{1}{2} a^2 \sin \phi \cos \phi$.

Expressed in terms of x , the area $OMPB$

$$= \int_0^a \sqrt{(a^2 - x^2)} dx = \frac{1}{2} a^2 \sin^{-1} x/a + \frac{1}{2} x \sqrt{(a^2 - x^2)}.$$

To find the second integral (ii.), substitute

$$x = a \cos \theta, \text{ then the angle } AOP = \theta;$$

and $y = a \sin \theta, dx = -a \sin \theta d\theta.$

Therefore the area PMA

$$\begin{aligned} &= - \int_{\theta}^0 a^2 \sin^2 \theta d\theta = \frac{1}{2} a^2 \int_0^{\theta} (1 - \cos 2\theta) d\theta \\ &= \frac{1}{2} a^2 \theta - \frac{1}{2} a^2 \sin \theta \cos \theta; \end{aligned}$$

the difference between $\frac{1}{2} a^2 \theta$, the area of the sector OAP , and $\frac{1}{2} a^2 \sin \theta \cos \theta$, the area of the triangle OMP .

Expressed in terms of x , the area PMA

$$= \int_a^0 \sqrt{(a^2 - x^2)} dx = \frac{1}{2} a^2 \cos^{-1} x/a - \frac{1}{2} x \sqrt{(a^2 - x^2)}.$$

Therefore the area of the quadrant OAB

$$= \int_0^a \sqrt{(a^2 - x^2)} dx = \frac{1}{4} \pi a^2;$$

and the area of the whole circle is πa^2 .

For the centroid of the area $OMPB$

$$\bar{x}A = \int_0^a x \sqrt{(a^2 - x^2)} dx = \frac{1}{3} a^3 - \frac{1}{3} (a^2 - x^2)^{\frac{3}{2}},$$

$$\bar{y}A = \int_0^a \frac{1}{2} (a^2 - x^2) dx = \frac{1}{2} a^2 x - \frac{1}{6} x^3.$$

For the centroid of the area PMA

$$\bar{x}A = \int_a^0 x \sqrt{(a^2 - x^2)} dx = \frac{1}{3} (a^2 - x^2)^{\frac{3}{2}} = \frac{1}{3} y^3,$$

$$\bar{y}A = \int_a^0 \frac{1}{2} (a^2 - x^2) dx = \frac{1}{2} a^2 (a - x) - \frac{1}{6} (a^3 - x^3).$$

Therefore, for the quadrant OAB , $\bar{x} = \bar{y} = 4a/3\pi$.

If we take A as the origin and $AM = x, MP = y$; then

$y = \sqrt{(2ax - x^2)}$; and the area $AMP = \int_0^a \sqrt{(2ax - x^2)} dx$;

which is reduced to the above by putting $x = a \cos \theta = 2a \sin^2 \frac{1}{2} \theta$, and then $AOP = \theta$.

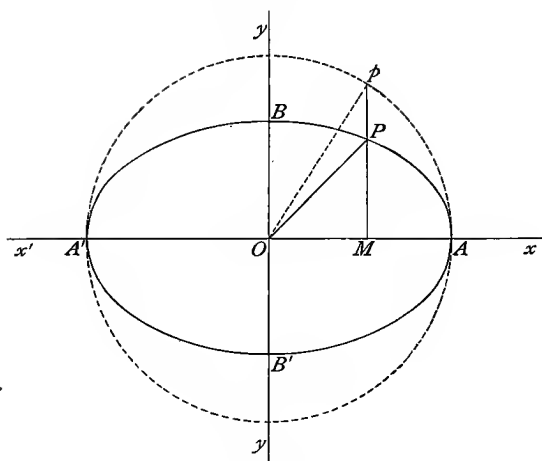


Fig. 22

51. The equation of the ellipse (fig. 22) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ or } y = \frac{b}{a} \sqrt{(a^2 - x^2)};$$

and therefore the area of a part of the ellipse is b/a of the corresponding part of the auxiliary circle cut off by the same ordinate, θ being called the *excentric* angle, and ϕ the complementary excentric angle, so that $\theta + \phi = \frac{1}{2}\pi$.

Thus the sector OAP of the ellipse $= \frac{1}{2}ab\theta$, and the sector $OPB = \frac{1}{2}ab\phi$, the quadrant of the ellipse OAB being $= \frac{1}{4}\pi ab$; also the triangle OMP

$$= \frac{1}{2}ab \sin \theta \cos \theta = \frac{1}{2}ab \cos \phi \sin \phi.$$

The ellipse APA' may be considered the *projection*, or shadow of the circle ApA' when turned through an angle α about the line AA' , thrown by rays of light perpendicular to the plane of the paper; so that OM is unaltered and $MP = Mp \cos \alpha = Mp \times b/a$.

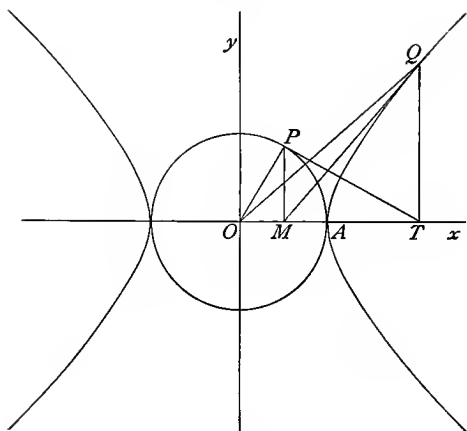


Fig. 23

52. *The quadrature of the hyperbola and its conjugate.*

In the rectangular hyperbola AQ (fig. 23)

$$x^2 - y^2 = a^2, \text{ or } y = \sqrt{(x^2 - a^2)};$$

and therefore the area ATQ

$$= \int_a^x \sqrt{(x^2 - a^2)} dx.$$

To find this integral by the hyperbolic functions (§ 34), substitute $x = a \cosh u$; then $y = a \sinh u$; $dx = a \sinh u du$.

Therefore the area ATQ

$$= \int_0^u a^2 \sinh^2 u du = \frac{1}{2} a^2 \int (\cosh 2u - 1) du$$

$$= \frac{1}{2} a^2 \left(\frac{1}{2} \sinh 2u - u \right) = \frac{1}{2} a^2 \cosh u \sinh u - \frac{1}{2} a^2 u,$$

of which the triangle $OTQ = \frac{1}{2} a^2 \sinh u \cosh u$, and therefore the sector $OAQ = \frac{1}{2} a^2 u$ (§ 34).

Expressed in terms of x , the area ATQ

$$= \int_a^x \sqrt{(x^2 - a^2)} dx$$

$$= \frac{1}{2} x \sqrt{(x^2 - a^2)} - \frac{1}{2} a^2 \cosh^{-1} x/a$$

$$= \frac{1}{2} x \sqrt{(x^2 - a^2)} - \frac{1}{2} a^2 \log \{x + \sqrt{(x^2 - a^2)}\} / a.$$

To find the integral by the circular functions, substitute $x = a \sec \theta$; then $y = a \tan \theta$, and the angle $AOP = \theta$; and (§ 34) $\theta = \text{gd } u$, or $\text{amh } u$; while $u = \log(\sec \theta + \tan \theta)$.

Then the area ATQ

$$\begin{aligned} &= a^2 \int_0^{\theta} \tan^2 \theta \sec \theta d\theta, \\ &= \frac{1}{2} a^2 \sec \theta \tan \theta - \frac{1}{2} a^2 \log(\sec \theta + \tan \theta) \\ &= \frac{1}{2} x \sqrt{(x^2 - a^2)} - \frac{1}{2} a^2 \log \{x + \sqrt{(x^2 - a^2)}\} / a, \end{aligned}$$

as before.

If we take A as the origin, and put $AT = x$, then

$$TQ = y = \sqrt{(2ax + x^2)},$$

and the area $ATQ = \int_0^x \sqrt{(2ax + x^2)} dx,$

which is reduced by the substitution

$$x = 2a \sinh^2 \frac{1}{2} u, \quad 2a + x = 2a \cosh^2 \frac{1}{2} u;$$

then $y = a \sinh u,$

and $dx = 2a \sinh \frac{1}{2} u \cosh \frac{1}{2} u du = a \sinh u du;$

so that the area

$$\begin{aligned} ATQ &= a^2 \int_0^u \sinh^2 u du = \frac{1}{2} a^2 \int (\cosh 2u - 1) du \\ &= \frac{1}{2} a^2 \cosh u \sinh u - \frac{1}{2} a^2 u, \end{aligned}$$

as before.

53. In the conjugate rectangular hyperbola (fig. 24),

$$y^2 - x^2 = a^2, \text{ or } y = \sqrt{(a^2 + x^2)}.$$

Then the area $OMRB = \int \sqrt{(a^2 + x^2)} dx,$

which is reduced to $a^2 \int \cosh^2 v dv$ by substituting $x = a \sinh v,$

$y = a \cosh v$, and then the sector $OBR = \frac{1}{2} a^2 v$; or the inte-

gral is reduced to $a^2 \int \sec^3 \phi d\phi$ by substituting $x = a \tan \phi,$

$y = a \sec \phi$, and then the angle $BOP = \phi = \text{gd } v$.

Therefore the area $OMRB$

$$\begin{aligned} &= \int_0^x \sqrt{a^2 + x^2} dx \\ &= \frac{1}{2}x\sqrt{a^2 + x^2} + \frac{1}{2}a^2 \sinh^{-1} x/a \\ &= \frac{1}{2}x\sqrt{a^2 + x^2} + \frac{1}{2}a^2 \log \{ \sqrt{a^2 + x^2} + x \} / a, \end{aligned}$$

and its centroid is given by

$$\bar{x}A = \int_0^x x\sqrt{a^2 + x^2} dx = \frac{1}{3}(a^2 + x^2)^{\frac{3}{2}} - \frac{1}{3}a^3,$$

$$\bar{y}A = \int_0^x \frac{1}{2}(a^2 + x^2) dx = \frac{1}{2}a^2x + \frac{1}{6}x^3.$$

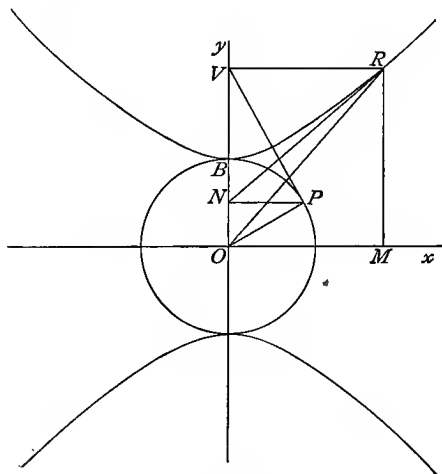


Fig. 24

54. The corresponding properties of the hyperbola

$$(x/a)^2 - (y/b)^2 = 1,$$

and of its conjugate hyperbola,

$$(y/b)^2 - (x/a^2) = 1,$$

are obtained immediately by projecting the preceding figures orthogonally on a plane parallel to Ox (fig. 14), that is by turning the figure through a certain angle α

about the line AA' , and then throwing its shadow by parallel rays (of sunlight) perpendicular to the plane of the paper; and then the coordinates of any point Q on the hyperbola are $x = a \cosh u$, $y = b \sinh u$,

so that $x/a + y/b = \exp u$;

and the sectorial area OAQ (fig. 14)

$$= \frac{1}{2}abu = \frac{1}{2}ab \log(x/a + y/b);$$

or we may substitute $x = a \sec \theta$, $y = b \tan \theta$.

The coordinates of any point P on the conjugate hyperbola are $x = a \sinh v$, $y = b \cosh v$,

where the sectorial area OBP (fig. 24)

$$= \frac{1}{2}abv = \frac{1}{2}ab \log(x/a + y/b);$$

or we may substitute $x = a \tan \phi$, $y = b \sec \phi$.

55. *The Quadrature of the Cycloid.*

Referring to fig. 9 of the cycloid (§ 21), and using θ as the independent variable, the area OMP

$$\begin{aligned} &= \int_0^x y dx = \int_0^\theta y \frac{dx}{d\theta} d\theta = \int_0^\theta a^2 (\text{vers } \theta)^2 d\theta \\ &= a^2 \int_0^\theta \{1 - 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta)\} d\theta \\ &= a^2 (\frac{3}{2}\theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta); \end{aligned}$$

and when the upper limit of integration is made 2π , corresponding to a complete revolution of the wheel, the area $OAB = 3\pi a^2 = 3$ times the area of the rolling circle.

This quadrature may be proved in an elementary manner (Roberval, 1634) by noticing that if NP is produced to meet the cycloid again in P' , then when the moving point has moved from P to P' the circle will have rolled through an angle $2\pi - \theta$ from O , so that

$$NP' = a(2\pi - \theta - \sin \theta), \text{ and } PP' = 2a(\pi - \theta + \sin \theta).$$

Again if $N'QQ'$ is drawn parallel to NPP' at an equal distance on the other side of the straight line Cc described by the centre of the wheel, meeting Oy in N' ,

then $QQ' = 2a(\theta + \sin \theta)$; so that $PP' + QQ' = 2\pi a + 4a \sin \theta$; and thus as PP' , QQ' recede from Cc at the same rate, they sweep out an area equal to $2\pi a \times NN' +$ the corresponding area cut out from the circle.

Therefore the whole area $OAB = 2\pi a^2 +$ area of the circle $= 3\pi a^2 = 3$ times the area of the rolling circle.

Examples.—Draw the following curves, and denoting the upper half of the area to the right of the axis of y by A and the coordinates of its centroid by \bar{x} , \bar{y} , prove that in

$$(1) \quad y^2 = ax - x^2, \quad A = \frac{1}{8}\pi a^2, \quad \bar{x} = \frac{1}{2}a, \quad \bar{y} = 2a/3\pi.$$

$$(2) \quad ay^2 = ax^2 - x^3, \quad A = \frac{4}{15}a^2, \quad \bar{x} = \frac{4}{7}a, \quad \bar{y} = \frac{5}{32}a.$$

$$(3) \quad a^2y^2 = ax^3 - x^4, \quad A = \frac{1}{16}\pi a^2, \quad \bar{x} = \frac{5}{8}a, \quad \bar{y} = 2a/5\pi.$$

$$(4) \quad xy^2 = a^3 - a^2x, \quad A = \frac{1}{2}\pi a^2, \quad \bar{x} = \frac{1}{4}a, \quad \bar{y} = \infty.$$

$$(5) \quad a^4y^2 = b^2x^2(a^2 - x^2), \text{ or } \sin^{-1}2y/b = 2 \sin^{-1}x/a, \\ A = \frac{1}{3}ab, \quad \bar{x} = \frac{3}{16}\pi a, \quad \bar{y} = \frac{1}{5}b.$$

$$(6) \quad x^4 - 2axy^2 + y^4 = 0, \quad A = \frac{1}{8}\sqrt{2}\pi a^2, \quad \bar{x} = \frac{1}{2}a, \quad \bar{y} = \frac{4}{3}\sqrt{2}a/\pi.$$

$$(7) \quad \text{In } (y-x)^2 = a^2 - x^2, \quad A = \frac{1}{2}\pi a^2, \quad \bar{x} = \bar{y} = 4a/3\pi;$$

where A now denotes the area of the curve to the right of the axis of y .

$$(8) \quad \text{In } x^4 - 2ax^2y + a^2(x^2 + y^2) - a^4 = 0, \\ A = \frac{1}{2}\pi a^2, \quad \bar{x} = 4a/3\pi, \quad \bar{y} = \frac{1}{2}a.$$

$$(9) \quad x^4 + 2x^2y^2 + 4ax^2y - 2a^2(x^2 - y^2 + 2ay) + a^4 = 0, \\ A = \pi a^2(2 - \frac{5}{4}\sqrt{2}), \text{ and determine } \bar{x} \text{ and } \bar{y}.$$

*(10) Prove that the area of a loop of the curve

$$(x^2 - na^2)^2 + (y^2 - a^2)^2 = a^4$$

$$\text{is } \frac{1}{2}\sqrt{2}a^2\{(n+1)\cot^{-1}\sqrt{n} - (n-1)\coth^{-1}\sqrt{n}\}.$$

56. *Quadrature with Polar Coordinates.*

Let $r=f\theta$ be the polar equation of a curve BPp (fig. 25); then if $xOP=\theta$, $OP=r=f\theta$.

Let the sectorial area OBP enclosed by the curve BP , a fixed initial vector OB , and the variable vector OP be denoted by A .

If ΔA and Δr denote the increments of A and r

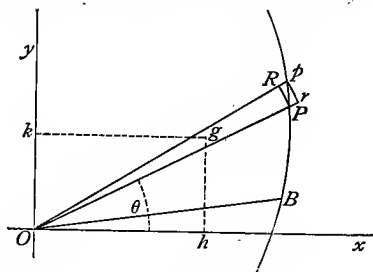


Fig. 25

corresponding to the increment $\Delta\theta$ of θ , and if $POp=\Delta\theta$, then $xOp=\theta+\Delta\theta$, $Op=r+\Delta r=f(\theta+\Delta\theta)$, and the area $OPp=\Delta A$.

Drawing the circular arcs PR , pr with centre O , the sectorial area OPp is seen to be intermediate to the circular sectors OPR and Opr ; or ΔA lies between $\frac{1}{2}r^2\Delta\theta$ and $\frac{1}{2}(r+\Delta r)^2\Delta\theta$, since the circular sector $OPR=\frac{1}{2}r^2\Delta\theta$, and the circular sector $Opr=\frac{1}{2}(r+\Delta r)^2\Delta\theta$; and therefore $\Delta A/\Delta\theta$ lies between $\frac{1}{2}r^2$ and $\frac{1}{2}(r+\Delta r)^2$.

Proceeding to the limit, by making $\Delta\theta$ indefinitely small,

$$dA/d\theta = \frac{1}{2}r^2.$$

$$\text{Therefore } A = \int_a^\theta \frac{1}{2}r^2 d\theta = \frac{1}{2} \int_a^\theta (f\theta)^2 d\theta,$$

if the angle $xOB=\alpha$; this is the formula to be employed with polar coordinates for the quadrature of a sector such as OBP .

The revolving radius OP , starting from OB , sweeps out the area OBP (the fluent), and makes the area grow at the rate $\frac{1}{2}r^2 d\theta/dt$ per unit of time t .

Also if \bar{x} , \bar{y} denote the coordinates of the centroid of the area OBP , then since g the centroid of the element OPp is ultimately in OP at a distance $\frac{2}{3}r$ from O , because OPp may be considered ultimately a triangle, therefore

$$\bar{x}A = \int_a^{\frac{2}{3}} r \cos \theta \frac{1}{2} r^2 d\theta = \frac{1}{3} \int_a^{\frac{2}{3}} r^3 \cos \theta d\theta,$$

$$\bar{y}A = \int_a^{\frac{2}{3}} r \sin \theta \frac{1}{2} r^2 d\theta = \frac{1}{3} \int_a^{\frac{2}{3}} r^3 \sin \theta d\theta,$$

where $r = f\theta$.

Examples.

- (1) Prove geometrically that $x \frac{dy}{dt} - y \frac{dx}{dt} = r^2 \frac{d\theta}{dt}$.
- (2) Find the area of a circle when its equation is given in the form $r = 2a \cos \theta$.

$$\begin{aligned} \text{(Here } A &= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{1}{2} r^2 d\theta = 2a^2 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^2 \theta d\theta \\ &= a^2 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (1 + \cos 2\theta) d\theta = \pi a^2.) \end{aligned}$$

- (3) Find the area and the centroid of a loop of the curve $r = a \cos 2\theta$.

$$\text{Answer : } A = \frac{1}{8} \pi a^2, \bar{x} = 128 \sqrt{2} a / 105 \pi.$$

- (4) Find the area and the centroid of a loop of the curve
 - (i.) $r = a \cos n\theta$, (ii.) $r = b \sin n\theta$,
 - (iii.) $r = a \cos n\theta + b \sin n\theta$ (fig. 26).

(*Solution.*—We must divide the right angle of the quadrants into n equal parts; and now in (i.) $r = a \cos n\theta$, the loop bounded by the lines $\theta = \pm \frac{1}{2}\pi/n$ may be taken (fig. 26); and then

$$\begin{aligned}
 A &= \int_{-\frac{1}{2}\pi/n}^{\frac{1}{2}\pi/n} \frac{1}{2}a^2 \cos^2 n\theta d\theta = \frac{1}{2}a^2 \int_0^{\frac{1}{2}\pi/n} (1 + \cos 2n\theta) d\theta \\
 &= \frac{1}{2}a^2 \left(\theta + \frac{\sin 2n\theta}{2n} \right) \Big|_0^{\frac{1}{2}\pi/n} = \frac{1}{4}\pi a^2/n;
 \end{aligned}$$

so that the loop is half the circumscribing sector OEE' of the circle $r=a$, bounded by the radii $\theta = \pm \frac{1}{2}\pi/n$.

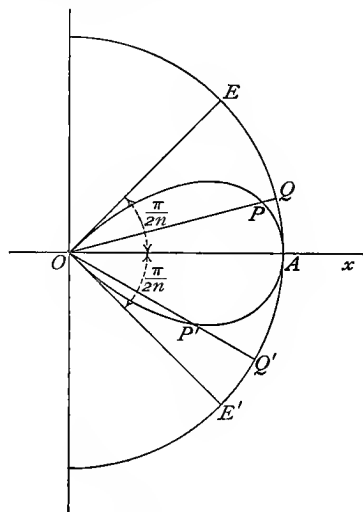


Fig. 26

This is geometrically obvious if we take vectors OP , OP' of the curve, such that $E'OP' = AOP = \theta$; then, $OP = a \cos n\theta$, $OP' = a \sin n\theta$, so that $OP^2 + OP'^2 = a^2$; and the sum of the areas swept out by OP and OP' is equal to the sector OAP or $OQ'E'$.

Again, for the centroid of the loop, $\bar{y} = 0$ by symmetry, and $\bar{x} = OG = \bar{r}$; while

$$\begin{aligned}
 \bar{x}A &= \int_{-\frac{1}{2}\pi/n}^{\frac{1}{2}\pi/n} \frac{1}{3}a^3 \cos^3 n\theta \cos \theta d\theta = \frac{1}{6}a^3 \int_0^{\frac{1}{2}\pi/n} (3 \cos n\theta + \cos 3n\theta) \cos \theta d\theta \\
 &= \frac{1}{12}a^3 \int_0^{\frac{1}{2}\pi/n} \{3 \cos(n-1)\theta + 3 \cos(n+1)\theta + \cos(3n-1)\theta + \cos(3n+1)\theta\} d\theta \\
 &= \frac{1}{12}a^3 \left\{ \frac{3 \sin(n-1)\theta}{n-1} + \frac{3 \sin(n+1)\theta}{n+1} + \frac{\sin(3n-1)\theta}{3n-1} + \frac{\sin(3n+1)\theta}{3n+1} \right\}_{\theta=0}^{\frac{1}{2}\pi/n} \\
 &= \frac{1}{12}a^3 \cos \frac{\pi}{2n} \left(\frac{3}{n-1} + \frac{3}{n+1} - \frac{1}{3n-1} - \frac{1}{3n+1} \right) = \frac{4a^3 n^3 \cos \frac{1}{2}\pi/n}{(n^2-1)(9n^2-1)}; \\
 \bar{x} = \bar{r} &= \frac{16an^4 \cos \frac{1}{2}\pi/n}{\pi(n^2-1)(9n^2-1)}.
 \end{aligned}$$

(ii.) In $r = b \sin n\theta$, we may take the loop bounded by $\theta = 0$ and $\theta = \pi/n$; and now, as before, $A = \frac{1}{4}\pi b^2/n$; while by symmetry $\bar{\theta} = \frac{1}{2}\pi/n$, and \bar{r} has the above value with b written for a .

(iii.) In $r = a \cos n\theta + b \sin n\theta$, it is convenient to introduce a subsidiary angle α , given by $\tan n\alpha = b/a$; and then $r = a \sec n\alpha \cos n(\theta - \alpha)$.

We may take the loop bounded by the lines $\theta = \alpha \pm \frac{1}{2}\pi/n$; so that

$$A = \int_{\alpha - \frac{1}{2}\pi/n}^{\alpha + \frac{1}{2}\pi/n} \frac{1}{2}a^2 \sec^2 n\alpha \cos^2 n(\theta - \alpha) d\theta = \frac{\pi a^2 \sec^2 n\alpha}{4n} = \frac{\pi(a^2 + b^2)}{4n};$$

while $\bar{\theta} = \alpha$, and \bar{r} has the above value with $\sqrt{(a^2 + b^2)}$ or $a \sec n\alpha$ substituted for a .

We thus see that the curves (i.), (ii.), and (iii.) are similar, but on a different scale, of a , b , and $\sqrt{(a^2 + b^2)}$, and differently orientated with respect to the origin O , which is the centre of similitude.)

(5) Find the area and centroid of the cardioid $r = a \operatorname{vers} \theta$.

$$\text{Answer:} \quad A = \frac{3}{2}\pi a^2, \quad \bar{x} = -\frac{5}{8}a.$$

(6) Prove that in the curve $r = a + b \cos n\theta$, where n is an integer, and $b < a$, the area $= \pi(a^2 + \frac{1}{2}b^2)$ and $\bar{x} = 0$; except when $n = 1$, and then $\bar{x} = b(a^2 + \frac{1}{4}b^2)/(a^2 + \frac{1}{2}b^2)$.

- (7) Find the centroid of a sector OAP of a circle of radius a (fig. 21).

Here $A = \frac{1}{2}a^2\theta$, and

$$\bar{x}A = \int_0^{\frac{1}{2}a^2\theta} \frac{2}{3}a \cos \theta \frac{1}{2}a^2 d\theta, \quad \bar{x} = \frac{2}{3}a (\sin \theta)/\theta,$$

$$\bar{y}A = \int_0^{\frac{1}{2}a^2\theta} \frac{2}{3}a \sin \theta \frac{1}{2}a^2 d\theta, \quad \bar{y} = \frac{2}{3}a (\text{vers } \theta)/\theta.$$

Similarly for the centroid of a sector OAP of an ellipse (fig. 22), $\bar{x} = \frac{2}{3}a(\sin \theta)/\theta$, $\bar{y} = \frac{2}{3}b(\text{vers } \theta)/\theta$, where θ is the excentric angle of the point P .

- (8) Find the centroid of a sector OAP of a hyperbola.

Here (fig. 23), $A = \frac{1}{2}abu$, $dA = \frac{1}{2}abdu$,

$$\bar{x}A = \int_0^{\frac{1}{2}abu} \frac{2}{3}a \cosh u \cdot \frac{1}{2}abdu, \quad \bar{x} = \frac{2}{3}a (\sinh u)/u,$$

$$\bar{y}A = \int_0^{\frac{1}{2}abu} \frac{2}{3}b \sinh u \cdot \frac{1}{2}abdu, \quad \bar{y} = \frac{2}{3}b (\text{versh } u)/u.$$

57. In finding with polar coordinates the area enclosed between two loops of the same curve or of different curves, or in finding the area between two *whorls* of a spiral curve, such as $r=a\theta$, the spiral of Archimedes, or $r=a^\theta$, the equiangular spiral (fig. 12), we must take r_1 to denote the vector to the outer branch and r_2 to the inner branch for the same value of θ ; and then the area will be given by $\int \frac{1}{2}(r_1^2 - r_2^2)d\theta$, taken between proper limits.

Thus for the area between the $(n+1)$ th and n th whorls of the spiral $r=a\theta$, we take

$$r_1 = 2n\pi a + a\theta, \quad r_2 = 2(n-1)\pi a + a\theta;$$

and the area

$$\int_{\theta}^{2\pi+\theta} \frac{1}{2}(r_1^2 - r_2^2)d\theta = 8n\pi^3 a^2 + 4\pi^2 \theta a^2 = 2\pi(r_1 - r_2)r_1;$$

so that the areas between the whorls increase in A.P.

Similarly it may be shown that the areas between the whorls of the equiangular spiral $r=a^\theta$ increase in G.P.

58. *Rectification of Curves.*

To rectify a curve we must determine ds/dx as a function of x , or ds/dy of y , or ds/dr of r , or $ds/d\theta$ of θ , or generally ds/dt as a function of t ; and then by integration determine s as a function of x , y , r , θ , or t .

The formulas of the Differential Calculus

$$\frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} \quad (\S 10), \quad \frac{ds^2}{dt^2} = \frac{dr^2}{dt^2} + \frac{r^2 d\theta^2}{dt^2} \quad (\S 23),$$

become in the Integral Calculus

$$s = \int \sqrt{\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)} dt, \quad s = \int \sqrt{\left(\frac{dr^2}{dt^2} + \frac{r^2 d\theta^2}{dt^2}\right)} dt;$$

$$\text{or} \quad s = \int \sqrt{dx^2 + dy^2}, \quad s = \int \sqrt{dr^2 + r^2 d\theta^2},$$

leaving the independent variable arbitrary; and it may be taken as x , or y , or r , or θ , or generally t .

Again if \bar{x} , \bar{y} are the coordinates of the centroid or centre of mass of a material curve or wire, of variable density σ per unit of length, then its mass $M = \int \sigma ds$; and

$$\bar{x}M = \int \sigma x ds, \quad \bar{y}M = \int \sigma y ds.$$

Examples.—Rectify the following curves—that is, find s in terms of x or y , or in polar coordinates in terms of θ or r . It is supposed that s is measured from the point where $x=0$, or $\theta=0$.

(1) The *semi-cubical* parabola $9ay^2 = 4x^3$.

$$\text{Here} \quad \frac{ds}{dx} = \sqrt{\left(1 + \frac{x}{a}\right)},$$

$$s = \int_0^x \sqrt{\left(1 + \frac{x}{a}\right)} dx = \frac{2}{3}a \left\{ \left(1 + \frac{x}{a}\right)^{\frac{3}{2}} - 1 \right\}.$$

- (2) Generally, in the curve $ay^{2n}=x^{2n+1}$, where n is an integer,

$$s=2na\left(\frac{2n}{2n+1}\right)^{2n}\int_1^{\infty}(z^2-1)^{n-1}z^2dz,$$

where $z=\sec \psi$, and $\tan \psi=\frac{dy}{dx}=\frac{2n+1}{2n}\left(\frac{x}{a}\right)^{1/2n}$.

And in the curve $y^{2n}=ax^{2n-1}$,

$$s=(2n-1)a\left(\frac{2n-1}{2n}\right)^{2n-1}\int_1^{\infty}(z^2-1)^{n-\frac{3}{2}}z^2dz,$$

where $z=\sec \omega$, and $\tan \omega=\frac{dx}{dy}=\frac{2n}{2n-1}\left(\frac{x}{a}\right)^{1/2n}$.

- (3) In $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, prove that $s=\frac{2}{3}a^{\frac{1}{3}}x^{\frac{2}{3}}$.

- (4) In $x^2+y^2=a^2$, $s=a \sin^{-1}x/a$;

and in $y^2=2ax-x^2$, $s=a \operatorname{vers}^{-1}x/a$.

- (5) In the catenary $y=a \cosh x/a$,

$$s=a \sinh x/a=\sqrt{(y^2-a^2)}=(\text{area } OMPA)/a.$$

- (6) In the tractrix AQ (fig. 15), $s=a \log a/y$.

- (7) In the catenary of equal strength $y=a \log \sec x/a$,

$$\cosh s/a=\sec x/a, \text{ or } x/a=\operatorname{gd} s/a.$$

- (8) In the cycloid (fig. 9), (rectified by Wren, 1658)

$$s=8a \operatorname{vers} \frac{1}{2}\theta=8a-4\sqrt{(4a^2-2ay)}.$$

- (9) With polar coordinates, prove that in the curve

$$r=a \cos \theta, \text{ or } a \sin \theta, s=a\theta.$$

- (10) In $r=a \sec \theta$, $s=a \tan \theta=\sqrt{(r^2-a^2)}$.

- (11) In the cardioid $r=a \operatorname{vers} \theta$, $s=4a \operatorname{vers} \frac{1}{2}\theta$.

- (12) In the parabola $r=2a/(1+\cos \theta)=a \sec^2 \frac{1}{2}\theta$,

$$\begin{aligned} s &= a \int_0^{\frac{1}{2}\theta} \sec^3 \frac{1}{2}\theta d\theta = a \sec \frac{1}{2}\theta \tan \frac{1}{2}\theta + a \log(\sec \frac{1}{2}\theta + \tan \frac{1}{2}\theta) \\ &= YP + a \log(\sec \frac{1}{2}\theta + \tan \frac{1}{2}\theta) \text{ (fig. 10) (Ex. 2. vi., § 32).} \end{aligned}$$

- (13) In the equiangular spiral $r=ae^{\theta \cot \alpha}$ (§ 30),

$$s=r \sec \alpha = Pt, \text{ if measured from the origin } O.$$

59. *The Volume and Surface of a Solid of Revolution.*

It has already been shown (§ 42) that for the volume V contained by the surface made by the revolution of the curve $y = fx$ round the axis of x and by planes perpendicular to the axis,

$$dV/dx = \pi y^2 = \pi (fx)^2, \text{ and } V = \pi \int y^2 dx = \pi \int (fx)^2 dx;$$

so that the volume V may be supposed built up of elementary discs of radius fx and thickness dx .

Also if \bar{x} is the abscissa of the centroid of the solid V , the centroid being in the axis of revolution,

$$\bar{x}V = \pi \int xy^2 dx = \pi \int x(fx)^2 dx.$$

Again, if S denotes the surface generated by the revolution of the curve $y = fx$, then the *fluent* S may be supposed generated by the motion of an expanding (or contracting) circumference of radius y , and therefore

$$\frac{dS}{dt} = 2\pi y \frac{ds}{dt}, \text{ or } \frac{dS}{dx} = 2\pi y \frac{ds}{dx}.$$

Therefore
$$S = 2\pi \int y \frac{ds}{dx} dx = 2\pi \int y ds,$$

and for the centroid of the surface

$$\bar{x}S = 2\pi \int xy \frac{ds}{dx} dx = 2\pi \int xy ds.$$

60. *Application to the Sphere, Spheroid, Paraboloid, and Cone.*

(i.) In the *sphere*, generated by the revolution of a circle (fig. 27) round the axis of x , $y^2 = a^2 - x^2$,

and therefore
$$V = \pi \int_0^a (a^2 - x^2) dx = \pi (a^2 x - \frac{1}{3} x^3),$$

$$\bar{x}V = \pi \int_0^a (a^2 x - x^3) dx = \pi (\frac{1}{2} a^2 x^2 - \frac{1}{4} x^4).$$

For a hemisphere, therefore, $V = \frac{2}{3}\pi a^3$, and $\bar{x} = \frac{3}{8}a$; and the volume of the complete sphere is $\frac{4}{3}\pi a^3$, $\frac{2}{3}$ of the circumscribing cylinder, or $\frac{1}{6}\pi d^3$, if d denotes the diameter.

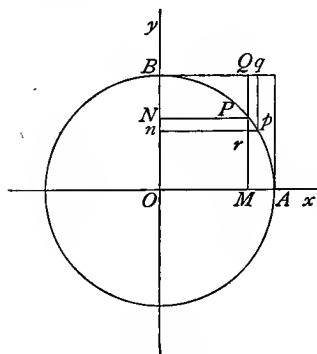


Fig. 27

Again, since

$$\frac{ds}{dx} = \frac{a}{y},$$

$$S = 2\pi \int_0^a y \frac{a}{y} dx = 2\pi ax;$$

$$\bar{x}S = \int 2\pi ax dx = \pi ax^2, \quad \bar{x} = \frac{1}{2}a.$$

(ii.) In the *spheroid*, generated by the revolution of an ellipse round the axis of x ,

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2);$$

and therefore

$$V = \pi \frac{b^2}{a^2} (a^2x - \frac{1}{3}x^3) = \pi ab^2 \left(\frac{x}{a} - \frac{1}{3} \frac{x^3}{a^3} \right);$$

and for the hemispheroid, $V = \frac{2}{3}\pi ab^2$, $\bar{x} = \frac{3}{8}a$.

If the spheroid is *prolate*, like a lemon, a is greater than b ; if *oblate*, like an orange, a is less than b .

(iii.) In the *paraboloid*, generated by the revolution of the parabola $y^2 = 2lx$,

$$V = \pi \int_0^x 2lx dx = \pi lx^2 = \frac{1}{2}\pi xy^2$$

$= \frac{1}{2}$ volume of the circumscribing cylinder; and $\bar{x} = \frac{2}{3}x$.

Also
$$S = 2\pi \int_0^y y \frac{ds}{dy} dy = 2\pi \int_0^y y \sqrt{\left(\frac{y^2}{l^2} + 1\right)} dy$$

$$= \frac{2}{3}\pi l^2 \left\{ \left(\frac{y^2}{l^2} + 1\right)^{\frac{3}{2}} - 1 \right\}.$$

(iv.) In the *cone*, generated by the revolution of the straight line $y = x \tan \alpha$, where α is the semi-vertical angle of the cone,

$$V = \pi \int_0^x x^2 \tan^2 \alpha dx = \frac{1}{3} \pi x^3 \tan^2 \alpha = \frac{1}{3} \pi x y^2$$

$$= \frac{1}{3} \text{ volume of the circumscribing cylinder ;}$$

and $\bar{x} = \frac{3}{4}x$.

Also $S = 2\pi \int_0^y y \operatorname{cosec} \alpha dy = \pi y^2 \operatorname{cosec} \alpha ;$

and for the centroid of the surface S , $\bar{x} = \frac{2}{3}x$.

61. If we refer to fig. 19, and compare the elements of volume swept out by the elements of area pn' and pm' when revolved about the axis Ox , we shall find these elements of volume are ultimately in the ratio of

$$\frac{2\pi xy dy}{\pi y^2 dx} = 2 \frac{xy dy}{y^2 dx},$$

or twice the ratio of the elements of area pn' and pm' .

Now, if the curve OP is given by the equation $y^n = p^{n-m} x^m$, this ratio of areas of pn' and pm' becomes m/n , and the ratio of the elements of volume is therefore $2m/n$; so that the area OMP is $n/(m+n)$ of the rectangle $OMPN$ (as before), while the volume swept out by the revolution of OMP about Ox is $n/(2m+n)$ of the volume of the cylinder swept out by $OMPN$.

When $m = n$, the curve OP is a straight line, and sweeps out a cone, and therefore the volume of the cone is one-third of the volume of the circumscribing cylinder; and when $2m = n$, the curve OP is a parabola with Ox for axis, and the volume of the paraboloid is one-half the volume of the circumscribing cylinder.

When $m = -n$, the curve becomes a rectangular hyperbola (fig. 20); and now the volume swept out by its

revolution round the asymptote Ox , to the right of the plane of MP , is equal to the cylinder swept out by the rectangle $OMPN$.

Similarly the volume swept out by the revolution of $y = a^x$ (fig. 11) to the left of MP is half again as great as that of the cone swept out by TMP .

In the sphere (fig. 27) the volumes swept out by the revolution round Ox of the elements of area Pn and Pq , in which $NP = x$, $MP = y$, $rp = \Delta x$, $Pr = -\Delta y$, are ultimately in the ratio

$$\frac{2\pi xy(-dy)}{\pi(a^2 - y^2)dx} = -2\frac{ydy}{x dx} = 2;$$

so that the volume of the sphere is thus $\frac{2}{3}$ of the circumscribing cylinder; while the elements of surface swept out by the elements Pp and Qq are ultimately in the ratio of $(2\pi y ds)/(2\pi a dx) = 1$; so that the whole surface of the sphere is equal to the curved surface of the circumscribing cylinder or the area of a circle of twice the radius (Archimedes).

62. Theorems of Pappus (or Guldin) (fig. 28).

Theorem I.—The volume V generated by the revolution of a closed plane curve of area A about an axis in its plane is equal to the volume of a cylinder of base A , and height equal to the circumference of the circle described by the centroid of the area A .

First consider the volume generated by the revolution of the area $ABDC$ in fig. 18; then

$$V = \pi \int y^2 dx; \text{ and } \bar{y}A = \frac{1}{2} \int y^2 dx,$$

if A denotes the area of $ABDC$ and \bar{y} the ordinate of G , the centroid of the area.

Therefore $V = 2\pi \bar{y}A$, which proves the theorem.

More generally, if A denotes the area of any closed plane curve not intersected by the axis of x (fig. 28), and if \bar{y} denotes the ordinate of its centroid G ; then supposing the area A to be cut up into elements of area dA at P ,

$$A = \int dA, \text{ and } \bar{y}A = \int y dA;$$

while $V = \int 2\pi y dA = 2\pi \bar{y}A.$

The volume generated by the revolution of the area A about the axis Oy will be $2\pi \bar{x}A$; and generally by the revolution about the line $x \cos \alpha + y \sin \alpha - p = 0$ will be

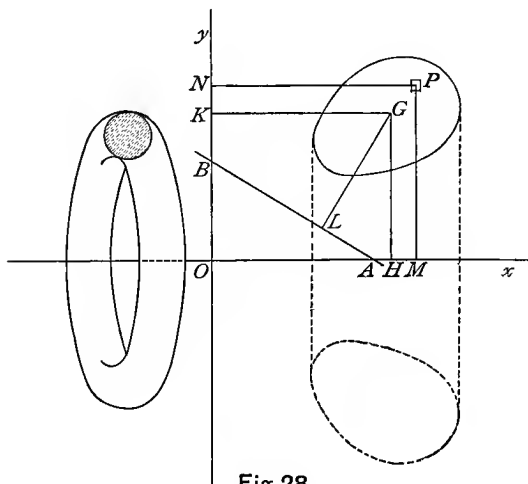


Fig. 28

$2\pi \times GL \times A = 2\pi(x \cos \alpha + y \sin \alpha - p)A$, where GL is the perpendicular on the line; so that if V_x , V_y denote the volumes generated by revolution about the axes Ox , Oy , and V about the line $x \cos \alpha + y \sin \alpha - p = 0$, then

$$V = V_y \cos \alpha + V_x \sin \alpha - 2\pi pA.$$

Theorem II.—The surface S generated by the revolution of a plane curve of length s about an axis in its plane is equal to a rectangle of base s and height equal to the circumference of the circle described by the centroid of the arc s .

For if \bar{y} denotes the ordinate of the centroid of the arc CD (fig. 18) or of the perimeter of the area in fig. 28, $\bar{y}s = \int y ds$ by § 58; also $S = 2\pi \int y ds$, so that $S = 2\pi \bar{y}s$, which proves Theorem II.

These theorems are of great use in Fortification and Civil Engineering, where the volume and surface of excavations have to be calculated.

Examples.

- (1) Prove that in the figure called the *anchor ring*, made by the revolution of a circle (of radius c) about an axis in the plane of the circle (at a distance a from the centre) (fig. 28),

$$V = 2\pi^2 ac^2, \quad S = 4\pi^2 ac.$$

- (2) Determine the ratio of the volumes and of the surfaces cut off from the anchor ring by the coaxial cylinder of radius a .
- (3) Prove that the volume of a *parabolic spindle* made by the revolution of a parabola about an ordinate is $\frac{8}{15}$ of the volume of the circumscribing cylinder.
- (4) Prove that the volume of the ogival head of an elongated projectile is $\frac{8}{15}$ of the cylindrical part of the same altitude, supposing the curve of the ogive is a parabola; and the volume of the rounded base is $\frac{2}{3}$ of the volume of the cylindrical part of the same altitude, supposing the curve of the base an ellipse.

- (5) Determine by Guldin's Theorems the centroid of the area and of the arc of a semicircle, knowing the volume and surface of a sphere, and the area and arc of a semicircle.
- (6) Prove that the volume swept out by the revolution about Oy of the circular segment PAP' cut off by the chord PMP (fig. 21) is $\frac{2}{3}\pi y^3 = \frac{1}{6}\pi \times PP^3$; this is the volume left of a sphere when a concentric cylindrical hole has been bored through it.
- Similarly the volume swept out by the revolution round Oy of the hyperbolic segment QAQ' is $\frac{1}{6}\pi \times QQ'^3$ (fig. 23).
- (7) Determine the volumes generated by the revolution about the coordinate axes of the curves in the examples of §§ 55, 56.
- (8) Prove that the volume and surface of a complete turn of the thread of a screw (fig. 7) is the same as that of the ring which would be swept out by the section of the thread made by a plane through the axis of the screw, when the screw makes a complete revolution without advancing.

63. In finding V with polar coordinates, the volume swept out by the sectorial element OPp (fig. 25) in revolving about the axis Ox is ultimately, by Guldin's Theorem, $2\pi \times \frac{2}{3}r \sin \theta \times \frac{1}{2}r^2 d\theta = \frac{2}{3}\pi r^3 \sin \theta d\theta$;

and
$$V = \int \frac{2}{3}\pi r^3 \sin \theta d\theta$$
;

and similarly
$$S = \int 2\pi r \sin \theta ds.$$

It is sometimes convenient to replace $\tan \theta$ by a single letter v , and then $y/x = v$; so that with the previous notation $A = \int \frac{1}{2}x^2 dv$, $\bar{x}A = \int \frac{1}{3}x^3 dv$, $\bar{y}A = \int \frac{1}{3}x^3 v dv$;

and
$$V = \int \frac{2}{3}\pi x^3 v dv.$$

Thus in the curve

$$x^3 + y^3 - 3axy = 0,$$

if we substitute $y = xv$, then $x = 3av/(1+v^3)$, and the area of the loop of the curve

$$= \int_0^\infty \frac{1}{2} x^2 dv = \frac{9}{2} a^2 \int_0^\infty \frac{v^2 dv}{(1+v^3)^2} = \frac{3}{2} a^2 \left(\frac{-1}{1+v^3} \right)_0^\infty = \frac{3}{2} a^2.$$

In a similar manner it may be shown that the area of the loop of $x^5 + y^5 - 5ax^2y^2 = 0$ is $\frac{5}{2}a^2$, and that the volume generated by the revolution round Ox of the loop of $x^5 + y^5 - 5ax^3y = 0$ is $\frac{2}{3}\pi a^3$.

64. *Moment of Inertia and Radius of Gyration.*

The *moment of inertia* of a body about an axis is defined to be, in the language of Indivisibles or Infinitesimals, *the sum of the products of the mass of each particle of the body and the square of its distance from the axis*; or in other words, in the language of Fluxions, *the space integral throughout the body of the product of the density and the square of the distance from the axis*.

The distance k from the axis at which the body, a fly-wheel for instance, may be supposed concentrated without altering its moment of inertia is called the *radius of gyration* of the body about the axis; and thus, if M denotes the mass of the body, then Mk^2 denotes its moment of inertia about the axis; and k^2 is equal to the moment of inertia divided by the mass.

When the body is homogeneous, we may suppose the density replaced by unity, and then the moment of inertia of the volume V is the space integral throughout the volume of the square of the distance from the axis, and may be denoted by Vk^2 , k denoting the radius of gyration about the axis.

Similarly we may define the moment of inertia of an area A as the surface integral over the area of the square of the distance from the axis, and denote it by Ak^2 ; and we may define the moment of inertia of a line of length l about an axis as the line integral of the square of the distance from the axis, and denote it by lk^2 , k denoting as before the radius of gyration.

A knowledge of the M.I. (moment of inertia) and radius of gyration of a body is requisite in certain mechanical problems; and as this quantity is difficult to find, except in some very elementary cases, unless we use the Integral Calculus, we shall discuss some simple cases here, in a manner analogous to that employed for determining areas and their centroids, as an illustration of the power of the Integral Calculus.

We shall first establish two fundamental theorems, which will simplify the subsequent operations.

Theorem I.—If k denotes the radius of gyration about an axis through the centroid or centre of gravity of a body, and k_1 denotes the radius of gyration about a parallel axis at a distance h , then $k_1^2 = k^2 + h^2$.

Take the origin O on the second axis and the plane of the paper perpendicular to the axis; then if m denotes the mass of a particle, and M the mass of the whole body,

$$M = \Sigma m, \text{ and } Mk_1^2 = \Sigma m(x^2 + y^2);$$

$$\text{while } Mk^2 = \Sigma m\{(x - \bar{x})^2 + (y - \bar{y})^2\},$$

if \bar{x} , \bar{y} , denote the coordinates of the centre of gravity; the symbol Σ denoting summation or integration throughout the body for the separate particles represented by m .

$$\text{But } \Sigma mx = M\bar{x}, \quad \Sigma my = M\bar{y},$$

$$\text{so that } Mk^2 = \Sigma m(x^2 + y^2) - \Sigma m(\bar{x}^2 + \bar{y}^2) = Mk_1^2 - Mh^2,$$

$$\text{or } k^2 = k_1^2 - h^2, \quad k_1^2 = k^2 + h^2.$$

From this theorem it follows that we need only calculate moments of inertia and radii of gyration about axes passing through the centroid or centre of gravity of a body; as by Theorem I. we pass immediately from the k^2 or Mk^2 about an axis through the centre of gravity to that about a parallel axis.

Theorem II.—If Ak_z^2 denotes the M.I. of a plane area A about an axis Oz perpendicular to the plane, and if Ak_x^2 , Ak_y^2 denote the M.I. about axes Ox , Oy at right angles to one another in the plane, then

$$Ak_z^2 = Ak_x^2 + Ak_y^2, \text{ or } k_z^2 = k_x^2 + k_y^2.$$

For if dA denotes an element of the area at the point x, y , then

$$Ak_x^2 = \int y^2 dA, \quad Ak_y^2 = \int x^2 dA, \quad Ak_z^2 = \int (x^2 + y^2) dA$$

so that $Ak_z^2 = Ak_x^2 + Ak_y^2$, or $k_z^2 = k_x^2 + k_y^2$.

A similar proof will hold where the superficial density over the area is supposed to be variable.

Examples.

- (1) Prove that $k^2 = \frac{1}{2}a^2$ for a circle (or cylinder) of radius a , about an axis through the centre perpendicular to the plane of the circle (or about the axis of the cylinder).

(Divide the circle into concentric circular elements of radius r , and breadth dr ;

then
$$Ak^2 = \int_0^a 2\pi r dr \times r^2 = \frac{1}{2}\pi a^4,$$

and
$$A = \pi a^2, \text{ therefore } k^2 = \frac{1}{2}a^2.$$

The same method holds for the cylinder.)

(2) Prove that $k^2 = \frac{1}{4}a^2$ for a circle about a diameter.

The k^2 about all diameters being the same, it is therefore by Theorem II. half the k^2 about the axis through the centre perpendicular to the plane of the circle.

Or independently, integrating with respect to x (fig. 21)

$$Ak^2 = \int_{-a}^a 2y dx \times x^2 = 4 \int_0^a x^2 \sqrt{(a^2 - x^2)} dx;$$

and substituting $x = a \sin \phi$,

$$\begin{aligned} Ak^2 &= 4 \int_0^{\frac{1}{2}\pi} a^4 \sin^2 \phi \cos^2 \phi d\phi = a^4 \int_0^{\frac{1}{2}\pi} (\sin 2\phi)^2 d\phi \\ &= \frac{1}{2} a^4 \int_0^{\frac{1}{2}\pi} (1 - \cos 4\phi) d\phi = \frac{1}{4} \pi a^4, \end{aligned}$$

and $A = \pi a^2$, so that $k^2 = \frac{1}{4}a^2$.

Similarly for an ellipse (fig. 22) about its axes, $k^2 = \frac{1}{4}b^2$ about OA , and $k^2 = \frac{1}{4}a^2$ about OB ; and $k^2 = \frac{1}{4}(a^2 + b^2)$ about an axis through O perpendicular to the plane.)

(3) Prove that $k^2 = \frac{1}{4}a^2 + \frac{1}{12}h^2$, for a cylinder of radius a and height h , about an axis through the centre and perpendicular to the axis of the cylinder.

(By Example (2) and Theorem I.,

$$Vk^2 = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \pi a^2 dx (\frac{1}{4}a^2 + x^2) = \pi a^2 h (\frac{1}{4}a^2 + \frac{1}{12}h^2);$$

and $V = \pi a^2 h$, so that $k^2 = \frac{1}{4}a^2 + \frac{1}{12}h^2$.

When $a = 0$, $k^2 = \frac{1}{12}h^2$, the result for a thin rod or material line; and when $h = 0$, $k^2 = \frac{1}{4}a^2$, the result for a thin disc.)

(4) Prove that $k^2 = \frac{2}{5}a^2$ for a sphere of radius a about a diameter, or for a spheroid about its axis.

$$\text{(Here } Vk^2 = \int_{-a}^a \pi y^2 dx \times \frac{1}{2}y^2 = \int_{-a}^a \frac{1}{2} \pi y^4 dx$$

and $y^2 = a^2 - x^2$; so that

$$Vk^2 = \frac{1}{2}\pi \int_{-a}^a (a^2 - x^2)^2 dx = \pi \int_0^a (a^4 - 2a^2x^2 + x^4) dx = \frac{8}{15}\pi a^5;$$

and $V = \frac{4}{3}\pi a^3$, so that $k^2 = \frac{8}{5}a^2$.)

- (5) Prove that $k^2 = \frac{3}{10}a^2$, for a cone of height h and radius of base a , about the axis; and $k^2 = \frac{3}{5}h^2 + \frac{3}{20}a^2$ about an axis through the vertex and perpendicular to the axis of the cone.

(About the axis of the cone

$$Vk^2 = \frac{1}{2}\pi \int_0^h y^4 dx = \frac{1}{2}\pi \frac{a^4}{h^4} \int_0^h x^4 dx = \frac{1}{10}\pi a^4 h;$$

and $V = \frac{1}{3}\pi a^2 h$, so that $k^2 = \frac{3}{10}a^2$,

About the perpendicular axis through the vertex

$$Vk^2 = \int_0^h \pi y^2 dx (x^2 + \frac{1}{4}y^2) = \frac{1}{3}\pi a^2 h (\frac{3}{5}h^2 + \frac{3}{20}a^2),$$

so that $k^2 = \frac{3}{5}h^2 + \frac{3}{20}a^2$.)

- (6) Prove that $k^2 = \frac{1}{3}a^2$ for a paraboloid of height h and radius of base a , about the axis; and that k^2 about a diameter of the base is $\frac{1}{6}(a^2 + h^2)$.

- (7) Prove that $k^2 = \frac{2}{21}d^2$ for the ogival pointed head of an elongated projectile of diameter d about the axis, the ogival part being half a parabolic spindle.

Prove also that, if the height of the ogival head is h , the distance of its centroid from the point is $\frac{1}{18}h$; and that, about a diameter of the base of the head, $k^2 = \frac{1}{7}h^2 + \frac{1}{21}d^2$.

- (8) Prove that $k^2 = \frac{1}{12}a^2$ for a line of length a about its middle point, or for a rectangle of length a and breadth b about a line in its plane through its centre perpendicular to the sides of length a .

- (9) Prove that $k^2 = \frac{1}{12}(a^2 + b^2)$ for this rectangle about an axis through its centre perpendicular to its plane.

Prove also that $k^2 = \frac{1}{12}(a^2 + b^2)$ for a right solid of edges a, b, c about an axis through its centre perpendicular to the edges a and b ; and that $k^2 = \frac{1}{3}(a^2 + b^2)$ about an edge of length c .

- (10) Prove that $k^2 = \frac{1}{2}h^2$ for a triangle of height h , about an axis in its plane through the vertex, parallel to the base; and thence $k^2 = \frac{1}{18}h^2$ for a parallel axis through the centroid of the triangle.

- (11) Prove that $k^2 = \frac{1}{2}h^2 + \frac{1}{24}a^2$ for an isosceles triangle of height h and base a about an axis through its vertex perpendicular to its plane; and thence that k^2 has the same value about the axis of a right prism standing on a regular polygonal base, a denoting the length of a side and h the radius of the inscribed circle of the regular polygon.

Deduce the value of k^2 about the axis of a circular cylinder.

- (12) Prove that if a solid is formed by the revolution through an angle θ of a plane area about an axis Ox in its plane (fig. 28) the moment of the volume about Ox is equal to the moment of inertia of the plane area about Ox multiplied by $2 \sin \frac{1}{2}\theta$.

Prove also that the moment of the curved surface generated is equal to the moment of inertia of the perimeter of the plane curve about Ox multiplied by $2 \sin \frac{1}{2}\theta$. (Prof. H. Hart, *Messenger of Mathematics*, vol. xiv., p. 100.)

**General Examples of Integration.*

- (1) Prove that the area of the curve $y(1+x^2)=1-x^3$, cut off by the axis of x , from $x=0$ to $x=1$, is

$$\frac{1}{4}\pi + \frac{1}{2}\log 2 - \frac{1}{2} = .631972.$$

- (2) Prove that the area, between the curve and its asymptotes, of $\frac{1}{x^2} - \frac{1}{y^2} = \frac{1}{a^2}$, or $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{a^2}$ is $4a^2$.

- (3) The area of $r^2 = a^2 \cos 2\theta$ (the *lemniscate*) is a^2 ; and for a loop, $\bar{x} = \frac{1}{8}\sqrt{2}\pi a$.

- (4) The area between the parabola $y^2 = ax$, and the circle $y^2 = 2ax - x^2$ is $\frac{1}{2}\pi a^2 - \frac{2}{3}a^2$, and $\bar{x} = a(\frac{1}{2}\pi - \frac{1}{3})/(\frac{1}{2}\pi - \frac{2}{3})$.

- (5) Find the area in the first quadrant contained by $y^2 = 4ax$, $x^2 + y^2 = 2ax$, $y = 2(x - 2a)$.

Express the area both when x is independent variable, and when y is independent variable.

- (6) A four-sided figure is formed by the three parabolas $y^2 - 9ax + 81a^2 = 0$, $y^2 - 4ax + 16a^2 = 0$, $y^2 - ax + a^2 = 0$; and by $y = 0$.

Prove that its area is $12a^2$, and is equal to the area enclosed by the chords of the arcs.

- (7) Prove that the area of each of the two equal and similar pieces bounded by the ellipse $(x/a)^2 + (y/b)^2 = 1$, and by the hyperbola $(x/a)^2 - (y/\beta)^2 = 1$, ($\alpha < a$), is

$$ab \sin^{-1} \frac{\beta \sqrt{(a^2 - \alpha^2)}}{\sqrt{(\alpha^2 \beta^2 + a^2 b^2)}} - \alpha \beta \sinh^{-1} \frac{b \sqrt{(a^2 - \alpha^2)}}{\sqrt{(\alpha^2 \beta^2 + a^2 b^2)}}.$$

- (8) Prove that, in the curve $x^m y^n = a^{m+n}$, the area between any two ordinates, the axis of x , and the curve is to the area between the curve and the corresponding radii vectores from the origin in the ratio $2n:m+n$; with oblique axes.

(9) Through any point on a hyperbola are drawn two straight lines parallel to the asymptotes to meet another hyperbola with the same asymptotes; show that the area intercepted between them and the curve is the same for all points.

(10) Prove that the area contained between a hyperbola, a tangent, and a line parallel to one of the asymptotes, which bisects the part of the tangent intercepted between the curve and the asymptote, is

$$\frac{1}{2}ab(\log 2 - \frac{5}{8}) = ab \times 0.034.$$

(11) Find the area of the curve $r^2 - 2ar \cos \theta + a^2 = b^2$.

(12) Prove that the area of a loop of the curve

(i.) $x^{2n} + y^{2n} = a^{2n}x^{n-1}y^{n-1}$ is $\frac{1}{4}\pi a^2/n$;

(ii.) $x^{2n+1} + y^{2n+1} = ax^n y^n$ is $\frac{1}{2}a^2/(2n+1)$.

(13) Trace, square, and rectify the curve

$$4(x^2 + y^2) - 3a^{\frac{4}{3}}x^{\frac{2}{3}} - a^2 = 0.$$

(14) Prove that in the curve

$$y = \frac{x^{n+1}}{2(n+1)a^n} + \frac{a^n}{2(n-1)x^{n-1}},$$

$$s = \frac{x^{n+1}}{2(n+1)a^n} - \frac{a^n}{2(n-1)x^{n-1}},$$

and $y^2 - s^2 = x^2/(n^2 - 1)$. (Newton.)

Discuss the particular case of $n = \frac{1}{2}$, when

$$9ay^2 = x(x - 3a)^2.$$

(15) Prove that in the curve

$$\sinh x/a \sinh y/a = 1, \text{ or } y/a = \log \coth \frac{1}{2}x/a, \\ s/a = \log \sinh x/a.$$

(16) Prove that s is an algebraical function of r in the

$$\text{curves } \frac{r}{a} = \left(\cos \frac{\theta}{2n} \right)^{2n}, \text{ or } \frac{r}{a} = \left(\sec \frac{\theta}{2n-1} \right)^{2n-1}.$$

(17) In the curves

$l/r = 1 + \sec a \cos (\theta \sin a)$, or $1 + \operatorname{sech} a \cosh (\theta \sinh a)$,
prove that

$l/p = \sec a + \cos (\theta \sin a)$, or $\operatorname{sech} a + \cosh (\theta \sinh a)$;

and $s = \frac{l \operatorname{cosec} a \sin (\theta \sin a)}{1 + \sec a \cos (\theta \sin a)}$, or $\frac{l \operatorname{cosech} a \sinh (\theta \sinh a)}{1 + \operatorname{sech} a \cosh (\theta \sinh a)}$.

(18) Prove that for the cycloid OAB revolving about the base OB (fig. 9), $V = 5\pi^2 a^3$, $S = \frac{64}{3}\pi a^2$.

(19) Find the perimeter and area of the curve

$$(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1,$$

by means of the substitution $x = a \cos^3 \theta$, $y = b \sin^3 \theta$;
find also the centroids of the quadrants, and the
volumes generated by the revolution of the curve
about the axes.

(Answer—Perimeter $= 4(a^2 + ab + b^2)/(a + b)$;

area $= \frac{3}{8}\pi ab$; $\bar{x} = \frac{\bar{y}}{b} = \frac{256}{315\pi}$; $V = \frac{32}{105}\pi a^2 b$, or $\frac{32}{105}\pi ab^2$.)

(20) Prove that for the catenary $y/a = \cosh x/a$ (fig. 15),
revolving about the axis of x ,

$$S = \pi(ax + sy), \quad V = \frac{1}{2}aS.$$

Prove that for the tractrix (fig. 15) revolving
about Ox , $S = 2\pi a(a - y)$, $V = \frac{1}{3}\pi(a^2 - y^2)^{\frac{3}{2}}$.

(21) Determine S , V , and the \bar{x} of V for the cardioid
 $r = a \operatorname{vers} \theta$.

CHAPTER III.

SUCCESSIVE DIFFERENTIATION.

65. *Notation of Successive Differential Coefficients.*

The operation of differentiation with respect to x being represented by the symbol $\frac{d}{dx}$, a second differentiation is represented by $\frac{d}{dx} \frac{d}{dx}$, which is written $\left(\frac{d}{dx}\right)^2$ or $\frac{d^2}{dx^2}$; and generally the operation of differentiating n times is represented by $\left(\frac{d}{dx}\right)^n$ or $\frac{d^n}{dx^n}$, so that the n^{th} d.c. of y with respect to x will be written $\left(\frac{d}{dx}\right)^n y$ or $\frac{d^n y}{dx^n}$.

Hitherto $\frac{dy^2}{dx^2}$ has been used generally for $\left(\frac{dy}{dx}\right)^2$; and $\left(\frac{dy}{dx}\right)^n$ may be written for $\frac{dy^n}{dx^n}$; but the difference between $\left(\frac{dy}{dx}\right)^n$ or $\frac{dy^n}{dx^n}$, meaning the n^{th} power of $\frac{dy}{dx}$, and $\frac{d^n y}{dx^n}$, meaning the n^{th} derivative of y , must be carefully observed.

If $y = fx$, and $\frac{dy}{dx}$ is denoted by $f'x$, then $\frac{d^2 y}{dx^2}$ is denoted by $f''x$, and generally $\frac{d^n y}{dx^n}$ by $f^n x$, or, more strictly, by $f^{(n)}x$.

Sometimes also the notation y', y'', y''', \dots and generally $y^{(n)}$ is employed to denote the successive derivatives of y with respect to x .

The successive derivatives of a function are required, among other purposes, in the expansion of a function by Taylor's Theorem, as explained in the next chapter.

66. Successive Differentiation of Rational Functions.

For instance (§ 4),

$$\frac{dx^m}{dx} = mx^{m-1};$$

therefore
$$\frac{d^2x^m}{dx^2} = m(m-1)x^{m-2},$$

$$\frac{d^3x^m}{dx^3} = m(m-1)(m-2)x^{m-3},$$

and generally

$$\frac{d^nx^m}{dx^n} = m(m-1)(m-2)\dots(m-n+1)x^{m-n}.$$

If m is a positive integer

$$\frac{d^mx^m}{dx^m} = m(m-1)(m-2)\dots 2 \cdot 1, \text{ denoted by } m!,$$

and all the higher derivatives vanish.

Generally to differentiate successively any rational function of x with respect to x , the function should first be resolved into its partial fractions (Smith, or Hall and Knight, *Algebra*), and then the n^{th} d.c. of a partial fraction $A/(x-a)$, written in the form $A(x-a)^{-1}$ will be

$$A(-1)^nn!(x-a)^{-n-1} \text{ or } A \frac{(-1)^nn!}{(x-a)^{n+1}};$$

and of a partial fraction

$$\frac{B}{(x-b)^m} \text{ will be } B \frac{(-1)^nm(m+1)\dots(m+n-1)}{(x-b)^{m+n}}.$$

67. To find the n^{th} d.c. of a partial fraction of the form

$$\frac{Px+Q}{(x-\alpha)^2+\beta^2},$$

corresponding to a quadratic factor in the denominator, suppose it resolved into its conjugate imaginary partial fractions of the form,

$$\frac{A+iB}{x-\alpha-i\beta} + \frac{A-iB}{x-\alpha+i\beta};$$

and then the n^{th} d.c. will be

$$\begin{aligned} & (-1)^n n! \left\{ \frac{A+iB}{(x-\alpha-i\beta)^{n+1}} + \frac{A-iB}{(x-\alpha+i\beta)^{n+1}} \right\} \\ &= (-1)^n n! \frac{(A+iB)(x-\alpha+i\beta)^{n+1} + (A-iB)(x-\alpha-i\beta)^{n+1}}{\{(x-\alpha)^2+\beta^2\}^{n+1}}, \end{aligned}$$

which is easily thrown into a real form.

Also (§ 29)
$$\frac{d \log (x-\alpha)}{dx} = \frac{1}{x-\alpha},$$

and therefore
$$\frac{d^n \log (x-\alpha)}{dx^n} = \frac{(-1)^{n-1} (n-1)!}{(x-\alpha)^n}.$$

68. *Successive Differentiation of Circular and Hyperbolic Functions.*

By § 17,
$$\frac{d \sin x}{dx} = \cos x = \sin(x + \tfrac{1}{2}\pi),$$

therefore
$$\frac{d^2 \sin x}{dx^2} = \sin(x + \tfrac{1}{2}2\pi),$$

and generally
$$\frac{d^n \sin x}{dx^n} = \sin(x + \tfrac{1}{2}n\pi).$$

Similarly,
$$\frac{d^n \sin(px+q)}{dx^n} = p^n \sin(px+q + \tfrac{1}{2}n\pi),$$

$$\frac{d^n \cos(px+q)}{dx^n} = p^n \cos(px+q + \tfrac{1}{2}n\pi).$$

Since

$$\frac{da^x}{dx} = a^x \log a \quad (\S 29),$$

therefore generally $\frac{d^n a^x}{dx^n} = a^x (\log a)^n$; and $\frac{d^n e^x}{dx^n} = e^x$.

$$\text{Also } (\S 33) \quad \frac{d^{2n} \sinh x}{dx^{2n}} = \sinh x, \quad \frac{d^{2n+1} \sinh x}{dx^{2n+1}} = \cosh x.$$

$$\frac{d^{2n} \cosh x}{dx^{2n}} = \cosh x, \quad \frac{d^{2n+1} \cosh x}{dx^{2n+1}} = \sinh x,$$

To differentiate any powers or products of the sine or cosine, circular or hyperbolic, we must express them as sines or cosines of the multiples of the argument, as in Integration, § 40.

Denoting by Fx any rational algebraical function of x ,

$$\text{then} \quad F\left(\frac{d}{dx}\right)e^{bx} = e^{bx} Fb, \quad F\left(\frac{d}{dx}\right)a^x = a^x F(\log a);$$

and supposing Fx divided into its even part fx^2 , and its odd part $x\phi x^2$ (§ 46), then

$$F\left(\frac{d}{dx}\right)_{\cos}^{\sin} mx = f(-m^2)_{\cos}^{\sin} mx + m\phi(-m^2)_{-\sin}^{\cos} mx,$$

$$F\left(\frac{d}{dx}\right)_{\sinh}^{\cosh} mx = f(m^2)_{\sinh}^{\cosh} mx + m\phi(m^2)_{\cosh}^{\sinh} mx;$$

theorems which are useful in the subject of *Differential Equations*, that is equations connecting functions and their derivatives.

In the following examples the formation of some simple differential equations will be shown, as the process will be instructive as a clue to the subsequent solution of these differential equations.

Examples.

(1) Differentiate n times

$$\frac{1, x, x^2, x^3}{x-a}, \quad \frac{1, x, x^2, \dots}{(x-a)(x-b)}, \quad \frac{1, x, x^2, \dots}{(x-a)(x-b)(x-c)}.$$

(Resolve into quotient and partial fractions.)

- (2) Differentiate n times, $\sin^2 x$, $\cos^3 x$, $\sin^4 x$, ..., $\cos mx \cos nx$, $\sin mx \cos nx$, $\sin mx \sin nx$; also the same functions with $\sinh x$ for $\sin x$ and $\cosh x$ for $\cos x$.

- (3) Prove that the differential equation $\frac{d^2x}{dt^2} + n^2x = 0$ is satisfied by $x = A \cos nt + B \sin nt$, or $x = a \cos(nt + \epsilon)$.

- (4) $\frac{d^2x}{dt^2} - n^2x = 0$ by $x = A \cosh nt + B \sinh nt$, or $x = ae^{nt} + be^{-nt}$.

- (5) $x^2y'' - (m+n-1)xy' + mny = 0$, by $y = Ax^m + Bx^n$.

- (6) $\frac{d^2x}{dt^2} + 2n \cos \beta \frac{dx}{dt} + n^2x = 0$,
by $x = ae^{-nt \cos \beta} \cos(nt \sin \beta + \epsilon)$.

- (7) $\frac{d^2x}{dt^2} - 2n \cosh \beta \frac{dx}{dt} + n^2x = 0$,
by $x = A \exp(n t e^\beta) + B \exp(n t e^{-\beta})$.

- (8) Given $y = \tanh^{-1}x$, $\frac{d^ny}{dx^n} = \frac{n! \{ (1+x)^n + (-1+x)^n \}}{(1-x^2)^n}$.

- (9) In the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,

$$\frac{d^2y}{dx^2} = \frac{\Delta}{(hx + by + f)^3} = \frac{\Delta}{\{(h^2 - ab)x^2 + 2(hf - bg)x + f^2 - bc\}^{\frac{3}{2}}};$$

where the discriminant $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$.

- (10) Given $y = e^{x \cos \alpha} \cos(x \sin \alpha)$,

$$\frac{dy}{dx} = e^{x \cos \alpha} \cos(x \sin \alpha + \alpha), \text{ and generally}$$

$$\frac{d^ny}{dx^n} = e^{x \cos \alpha} \cos(x \sin \alpha + n\alpha).$$

When $\alpha = \pi/n$, $\frac{d^ny}{dx^n} \pm y = 0$, as n is odd or even.

- (11) $y = e^{ax+b} \cos(px+q)$, and $\tan \alpha = b/a$ (Ex. 20, p. 79).

$$\frac{d^ny}{dx^n} = a^n (\sec \alpha)^n e^{ax+b} \cos(px+q+n\alpha),$$

$$(12) \quad y = \cosh ax \cos px,$$

$$\frac{d^n y}{dx^n} = a^n (\sec a)^n (\cosh ax \cos px \cos na - \sinh ax \sin px \sin na),$$

$$\text{or} = a^n (\sec a)^n (\sinh ax \cos px \cos na - \cosh ax \sin px \sin na),$$

as n is even or odd.

Given also $y = \sinh ax \sin px$, determine $d^n y/dx^n$.

$$(13) \quad \text{Given } y = x^p (A \cos nx + B \sin nx), \text{ prove that}$$

$$x^2 \frac{d^2 y}{dx^2} - 2px \frac{dy}{dx} + (n^2 x^2 + p \cdot p + 1)y = 0.$$

$$(14) \quad \text{Given } \alpha^n u^n = \sec n\theta, \quad \frac{d^2 u}{d\theta^2} + u = (n+1)\alpha^{2n} u^{2n+1}.$$

$$(15) \quad \frac{d^2 \log y}{dx^2} = \frac{1}{y} \frac{d^2 y}{dx^2} - \left(\frac{1}{y} \frac{dy}{dx} \right)^2 = \frac{y''}{y} - \left(\frac{y'}{y} \right)^2.$$

$$(16) \quad \text{Given } y = \frac{ax+b}{Ax+B}, \text{ prove that } \frac{y'''}{y'} - \frac{3}{2} \left(\frac{y''}{y'} \right)^2 = 0;$$

and given $z = (ay+b)/(Ay+B)$, prove that

$$\frac{z'''}{z'} - \frac{3}{2} \left(\frac{z''}{z'} \right)^2 = \frac{y'''}{y} - \frac{3}{2} \left(\frac{y''}{y'} \right)^2.$$

This function of y or z is called by Professor Cayley the *Schwartzian derivative*, and is denoted by him by (y, x) or (z, x) .

69. Leibnitz's Theorem.

We have already proved the rule for the differentiation of uv , the product of u and v , two given functions of x ;

$$\text{namely (§ 12)} \quad \frac{d(uv)}{dx} = \frac{du}{dx} v + u \frac{dv}{dx};$$

and Leibnitz's Theorem enables us to generalize this differentiation for any number of repetitions of the operation.

For differentiating again, each term on the right-hand side, being a product, gives rise to two terms; and taking care not to invert the order of u and v ,

$$\begin{aligned}\frac{d^2uv}{dx^2} &= \frac{d^2u}{dx^2}v + \frac{du}{dx} \frac{dv}{dx} + \frac{du}{dx} \frac{dv}{dx} + u \frac{d^2v}{dx^2} \\ &= \frac{d^2u}{dx^2}v + 2 \frac{du}{dx} \frac{dv}{dx} + u \frac{d^2v}{dx^2}.\end{aligned}$$

Again $\frac{d^3uv}{dx^3} = \frac{d^3u}{dx^3}v + 3 \frac{d^2u}{dx^2} \frac{dv}{dx} + 3 \frac{du}{dx} \frac{d^2v}{dx^2} + u \frac{d^3v}{dx^3};$

and $\frac{d^4uv}{dx^4} = \frac{d^4u}{dx^4}v + 4 \frac{d^3u}{dx^3} \frac{dv}{dx} + 6 \frac{d^2u}{dx^2} \frac{d^2v}{dx^2} + 4 \frac{du}{dx} \frac{d^3v}{dx^3} + u \frac{d^4v}{dx^4}.$

We now perceive the law for any number of differentiations, by analogy with the Binomial Theorem; and the law can be proved by Mathematical Induction.

For assume that

$$\begin{aligned}\frac{d^nuv}{dx^n} &= \frac{d^nu}{dx^n}v + n \frac{d^{n-1}u}{dx^{n-1}} \frac{dv}{dx} + \frac{n(n-1)}{1.2} \frac{d^{n-2}u}{dx^{n-2}} \frac{d^2v}{dx^2} + \dots \\ &+ \frac{n(n-1)}{1.2} \frac{d^2u}{dx^2} \frac{d^{n-2}v}{dx^{n-2}} + n \frac{du}{dx} \frac{d^{n-1}v}{dx^{n-1}} + u \frac{d^nv}{dx^n} \dots \dots \dots (1)\end{aligned}$$

Differentiating again, each term on the right-hand side of (1) gives rise to two terms, of which the second of one term coalesces with the first of the next term; so that

$$\begin{aligned}\frac{d^{n+1}uv}{dx^{n+1}} &= \frac{d^{n+1}u}{dx^{n+1}}v + (n+1) \frac{d^nu}{dx^n} \frac{dv}{dx} + \frac{(n+1)n}{1.2} \frac{d^{n-1}u}{dx^{n-1}} \frac{d^2v}{dx^2} + \dots \\ &+ \frac{(n+1)n}{1.2} \frac{d^2u}{dx^2} \frac{d^{n-1}v}{dx^{n-1}} + (n+1) \frac{du}{dx} \frac{d^nv}{dx^n} + u \frac{d^{n+1}v}{dx^{n+1}} \dots \dots (2)\end{aligned}$$

If therefore the law expressed by (1) holds for n , it holds when n is changed into $n+1$, as expressed in (2).

But the law holds when n is 1, 2, 3, 4, and therefore it holds when n is 5, 6, ... and generally when n is any positive integer. This law is called *Leibnitz's Theorem*.

Leibnitz's Theorem can be established by a *symbolical* proof, which is easily extended to the case of the differentiation of any number of factors; for if

$$y = uvw \dots$$

where u, v, w, \dots are functions of x , then

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx}vw \dots + u \frac{dv}{dx}w \dots + uv \frac{dw}{dx} \dots + \dots \\ &= (D_1 + D_2 + D_3 + \dots)uvw \dots, \end{aligned}$$

where D_1 represents the operation of differentiation on u only, D_2 on v , D_3 on w, \dots

Then, since these *operators* represented by D obey the same laws as algebraical quantities,

$$\frac{d^n y}{dx^n} = (D_1 + D_2 + D_3 + \dots)^n uvw \dots,$$

so that the coefficient of $\frac{d^p u}{dx^p} \frac{d^q v}{dx^q} \frac{d^r w}{dx^r} \dots$ is equal to the coefficient of $x^p y^q z^r \dots$ in the expansion of $(x + y + z + \dots)^n$ by the Multinomial Theorem.

70. By Leibnitz's Theorem,

$$\begin{aligned} \frac{d^n e^{ax} y}{dx^n} &= e^{ax} \left\{ a^n y + n a^{n-1} \frac{dy}{dx} + \frac{n(n-1)}{1 \cdot 2} a^{n-2} \frac{d^2 y}{dx^2} + \dots \right\} \\ &= e^{ax} \left\{ a^n + n a^{n-1} \frac{d}{dx} + \frac{n(n-1)}{1 \cdot 2} a^{n-2} \frac{d^2}{dx^2} + \dots \right\} y \\ &= e^{ax} \left(a + \frac{d}{dx} \right)^n y, \end{aligned}$$

with the *symbolical* notation.

Therefore, if Fx denotes a rational function of x ,

$$F\left(\frac{d}{dx}\right) e^{ax} y = e^{ax} F\left(a + \frac{d}{dx}\right) y,$$

and $F\left(\frac{d^2}{dx^2}\right) y \sin(px + q) = \sin(px + q) F\left(\frac{d^2}{dx^2} - p^2\right) y,$

theorems of great use in *Differential Equations*.

Examples on Leibnitz's Theorem.

(1) Differentiate n times,

$$x \sin x, x^2 \cosh 2x, (x \sin x)^3, x^m e^{ax}, x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y.$$

(2) Prove that, if

$$(i.) y = \sin^{-1} x, (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = 0; \text{ and, when } x=0,$$

$$y^{(2n)} = 0, y^{(2n+1)} = 1^2 \cdot 3^2 \cdot 5^2 \dots (2n-1)^2.$$

$$(ii.) y = (\sin^{-1} x)^2, (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - 2 = 0,$$

$$\text{and } (1-x^2) \frac{d^{n+2} y}{dx^{n+2}} - (2n+1)x \frac{d^{n+1} y}{dx^{n+1}} - n^2 \frac{d^n y}{dx^n} = 0.$$

$$\text{When } x=0, y^{(2n+1)} = 0, y^{(2n)} = 2 \cdot 2^2 \cdot 4^2 \dots (2n-2)^2.$$

$$(iii.) y = \sin(m \sin^{-1} x + a), (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0,$$

$$(1-x^2) \frac{d^{n+2} y}{dx^{n+2}} - (2n+1)x \frac{d^{n+1} y}{dx^{n+1}} + (m^2 - n^2) \frac{d^n y}{dx^n} = 0.$$

Determine $y^{(n)}$ when $x=0$. (Newton.)

$$(iv.) y = \tan^{-1} x/a, (a^2 + x^2) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = 0,$$

$$\text{and } (a^2 + x^2) \frac{d^{n+2} y}{dx^{n+2}} + 2(n+1)x \frac{d^{n+1} y}{dx^{n+1}} + n(n+1) \frac{d^n y}{dx^n} = 0.$$

$$\text{When } x=0, y^{(2n)} = 0, y^{(2n+1)} = (-1)^n 2n! a^{-2n}.$$

(3) Prove that the differential equation $\left(\frac{d}{dx} - a\right)^n y = 0$

is satisfied by $y = (C_0 + C_1 x + \dots + C_{n-1} x^{n-1}) e^{ax}$;

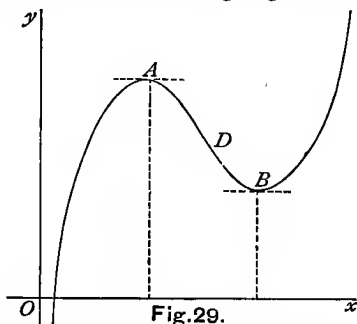
and $\left(\frac{d^2}{dx^2} \pm p^2\right)^n y = 0$ by

$$y = (C_0 + C_1 x + \dots + C_{n-1} x^{n-1}) (A_{\cosh}^{\cos} p x + B_{\sinh}^{\sin} p x).$$

71. *The Maximum and Minimum Values of a Function.*

One of the most useful applications of the Calculus is to the determination of the maximums or minimums of a function y or fx of a variable quantity x .

The subject has been already touched upon in § 5, where it has been shown that since dy/dx is positive if y increases with x , but dy/dx is negative if y diminishes as x increases, therefore $dy/dx=0$ at the turning points, such as A , where y from increasing begins to diminish, or as B , where y from diminishing begins to increase again.



At A , y is said to have a *maximum* value; and dy/dx is *diminishing* in passing through the value zero from a positive to a negative value, and therefore d^2y/dx^2 is negative.

At B , y is said to have a *minimum* value; and dy/dx is *increasing* through the value zero from a negative to a positive value, and d^2y/dx^2 is positive.

Thus to discover the maximum or minimum value of y , a function of x , we must first find the values of x which make $dy/dx=0$.

If one of these values makes d^2y/dx^2 negative, the corresponding value of y is a maximum; but if it makes d^2y/dx^2 positive, the value of y is a minimum.

But if d^2y/dx^2 is also zero, then a closer examination is required; and it may happen, as at C (fig. 31) that y has neither a maximum or minimum value, although dy/dx is zero.

Generally d^2y/dx^2 is zero when dy/dx is a maximum or minimum, that is where the road ABC is steepest, as at D ; such a point D is called a *point of inflexion* on the curve ABC , the curve crossing the tangent at the point, and changing its curvature from one way to the other; as seen in railway lines, on an S curve.

As x increases continuously, the maximums and minimums of y must occur alternately, because y after reaching a maximum must diminish again to a minimum before increasing again to a maximum.

It is advisable in the following examples to sketch the graph of the function whose maximums or minimums are required.

Examples.

(1) If $y = 2x^3 - 9x^2 + 12x - 3$,

(§ 5), the equation of the curve of fig. 29;

$$dy/dx = 6x^2 - 18x + 12 = 6(x-1)(x-2),$$

which vanishes, when $x=1$, or 2 ; and

$$d^2y/dx^2 = 6(x-2) + 6(x-1).$$

(i.) When $x=1$, $d^2y/dx^2 = -6$, and $y=2$, a maximum;

(ii) when $x=2$, $d^2y/dx^2 = 6$, and $y=1$, a minimum.

(2) $y = ax - x^2$, is a maximum $\frac{1}{2}a^2$, when $x = \frac{1}{2}a$ (§ 5).

(3) $y = x - x^3$, is a maximum and equal $\frac{2}{9}\sqrt{3}$, when $x = \frac{1}{3}\sqrt{3}$; a minimum $-\frac{2}{9}\sqrt{3}$, when $x = -\frac{1}{3}\sqrt{3}$.

(4) Prove that $y = x^3 - 3x^2 + 6x$ has no max. or min. value.

(5) If $y = (1+x^2)(7-x^3)^2$; when $x=0$, $y=49$, a minimum; $x=1$, $y=72$, a max.; $x = \sqrt[3]{7}$, $y=0$, a minimum.

(We find $dy/dx = 2x(1-x)(7-x^3)(7+4x+4x^2)$, which vanishes for real values of x only when $x=0$, 1, or $\sqrt[3]{7}$, arranging the values in ascending order.

As we merely require the sign of d^2y/dx^2 for the corresponding values of x , it is convenient to proceed as follows, and thereby avoid writing down superfluous terms:—

(i.) when $x=0$, $y=49$, and

$$d^2y/dx^2 = 2(1-x)(7-x^3)(7+4x+4x^2) + \text{terms which vanish when } x=0, = +98; \text{ and } y \text{ is a min.};$$

(ii.) when $x=1$,

$$d^2y/dx^2 = -2x(7-x^3)(7+4x+4x^2) + \dots \\ = -180; \text{ and therefore } y \text{ is a maximum};$$

(iii.) when $x=\sqrt[3]{7}$,

$$d^2y/dx^2 = -6x^2(1-x)(7+4x+4x^2) + \dots \\ = \text{a positive number, and } y \text{ is a minimum.})$$

(6) $y=(x-1)(x-2)/(x-3)$ is a max., $3-2\sqrt{2}$, when $x=3-\sqrt{2}$; a min., $3+2\sqrt{2}$, when $x=3+\sqrt{2}$.

Explain how the min. is greater than the max.

(7) If $y = \frac{x^2-x+1}{x^2+x-1}$; when $x=0$, $y=-1$, a maximum; when $x=2$, $y=\frac{3}{5}$, a minimum.

(8) Determine the maximum and minimum of

$$y = \frac{ax^2 + 2bx + c}{Ax^2 + 2Bx + C}$$

(This can be done algebraically, by solving this equation as a quadratic in x , and determining the limits of y from a consideration of the expression under the radical.

Then $(Ay-a)x^2 + 2(By-b)x + Cy-c = 0$; and solving this quadratic in x ,

$$x = \frac{-(By-b) \pm \sqrt{\{(By-b)^2 - (Ay-a)(Cy-c)\}}}{Ay-a}.$$

The turning points of y are given by

$$(Ay - a)(Cy - c) - (By - b)^2 = 0,$$

$$\text{or } (AC - B^2)y^2 - (Ac + aC - 2Bb)y + ac - b^2 = 0 \dots\dots\dots (1)$$

$$\text{and then } x = -(By - b)/(Ay - a),$$

$$\text{or } y = \frac{ax + b}{Ax + B} = \frac{bx + c}{Bx + C} \dots\dots\dots (2)$$

$$\text{so that } (Ab - aB)x^2 + (Ac - aC)x + Bc - bC = 0 \dots\dots\dots (3)$$

the transformation of (1) into (3) by the *homographic substitution* (2); obtainable also by putting $dy/dx = 0$.

The roots of (1) and (3) will be imaginary only when the roots of $Ax^2 + 2Bx + C = 0$ separate the roots of $ax^2 + 2bx + c = 0$, as will be seen on drawing the graph of

$$y = (ax^2 + 2bx + c)/(Ax^2 + 2Bx + C);$$

for instance, the graphs of

$$y = (x - 1)(x - 3)/(x - 2)(x - 4),$$

$$\text{or } y = (x - 1)(x - 2)/(x - 3)(x - 4).$$

In all other cases the roots of (1) and (3) are real; and denoting them by y_1, y_2 , and x_1, x_2 , then

$$y_1 - y = \frac{(Ay_1 - a)(x - x_1)^2}{Ax^2 + 2Bx + C}, \quad y - y_2 = \frac{(a - Ay_2)(x - x_2)^2}{Ax^2 + 2Bx + C}$$

and y_1 being the max., y_2 will be the min. value of y .

We now find

$$ax^2 + 2bx + c = p(x - x_1)^2 + q(x - x_2)^2, \\ Ax^2 + 2Bx + C = P(x - x_1)^2 + Q(x - x_2)^2;$$

$$\text{where } P = \frac{Ay_1 - a}{y_1 - y_2}, \quad Q = \frac{a - Ay_2}{y_1 - y_2}, \\ p = \frac{y_2(Ay_1 - a)}{y_1 - y_2}, \quad q = \frac{y_1(a - Ay_2)}{y_1 - y_2};$$

so that $p = Py_2, q = Qy_1$.

In constructing numerical examples, we may assign arbitrary integral values to x_1, x_2, P, Q, p, q ; and then integral values of A, B, C, a, b, c result, which make equations (1) and (3) have rational roots.

Thus $y = (x^2 + x + 1)/(x^2 - x + 1)$ may be written

$$y = \frac{(x-1)^2 + 3(x+1)^2}{3(x-1)^2 + (x+1)^2};$$

so that $x_1 = 1, x_2 = -1, P = 3, Q = 1, p = 1, q = 3$;
and $y_1 = 3, y_2 = \frac{1}{3}$.

Again y in Ex. 7 may be written

$$y = \{3x^2 + (x-2)^2\} / \{5x^2 - (x-2)^2\}.$$

(9) Determine the maximum and minimum of

$$r = a \cos \theta + b \sin \theta.$$

(Writing it $r = \sqrt{(a^2 + b^2)} \cos(\theta - \tan^{-1} b/a)$,

then $r = \sqrt{(a^2 + b^2)}$ a maximum, when $\theta = \tan^{-1} b/a$;

$r = -\sqrt{(a^2 + b^2)}$, a minimum, when $\theta = \pi + \tan^{-1} b/a$.)

(10) Supposing given currents C and C' produce deflexions α and α' in a tangent galvanometer, so that $\tan \alpha / \tan \alpha' = C/C'$; show how to make $\alpha - \alpha'$ a maximum.

(Here $\sin(\alpha - \alpha') = \frac{C - C'}{C + C'} \sin(\alpha + \alpha')$;

and therefore $\alpha - \alpha'$ is a maximum when $\alpha + \alpha' = \frac{1}{2}\pi$.)

(11) Determine the maximum and minimum of y , when x and y are connected by the implicit relation (§ 13)

$$x^3 - 3axy + y^3 = 0.$$

(Forming the first derived equation

$$3x^2 - 3ay - 3ax \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0;$$

then $dy/dx = 0$, if $x^2 - ay = 0$.

Combining this with the implicit relation we obtain

$$x = a\sqrt[3]{2}, y = a\sqrt[3]{4}.$$

To find the corresponding value of d^2y/dx^2 , form the *second* derived equation by differentiating the first derived equation; but omitting terms involving dy/dx , since they vanish; therefore

$$6x - 3ax \frac{d^2y}{dx^2} + 3y^2 \frac{d^2y}{dx^2} = 0,$$

or
$$\frac{d^2y}{dx^2} = \frac{2x}{ax - y^2} = -\frac{2}{a},$$

so that the corresponding value of y is a maximum.

Similarly, by differentiating with respect to y , equating dx/dy to zero, and examining the sign of d^2x/dy^2 , we find that when $y = a\sqrt[3]{2}$, x has the maximum value of $a\sqrt[3]{4}$.

These considerations are sometimes useful in drawing a curve whose equation is given as above.)

(12) If $xy(x - y) = 2a^3$, determine the maximum and minimum values of x and y ; also if $x^4 + y^4 - 4a^2xy = 0$.

(13) Determine the greatest rectangle which can be inscribed in a given isosceles (or scalene) triangle.

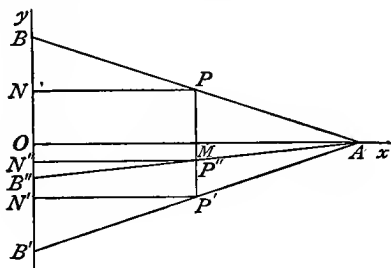


Fig.30.

(Let ABB' be the given isosceles triangle (fig. 30) and let a denote the altitude OA and $2b$ the base BB' ; and x the height and $2y$ the breadth of the inscribed rectangle $PNN'P'$; or $PNN''P''$ in the scalene triangle ABB'').

Then $x/a + y/b = 1$, since P lies on the straight line AB .

Also, if u denotes the area of the rectangle $PNN'P'$,

$$u = 2xy = 2xb\left(1 - \frac{x}{a}\right) = 2b\left(x - \frac{x^2}{a}\right),$$

and
$$\frac{du}{dx} = 2b\left(1 - 2\frac{x}{a}\right) = 0,$$

when $x = \frac{1}{2}a$, $y = \frac{1}{2}b$; and then $\frac{d^2u}{dx^2} = -4\frac{b}{a}$;

so that $u = \frac{1}{2}ab$, a maximum.

This is the problem of cutting the greatest rectangular log from a triangular log; at least half the material is wasted.)

(14) Determine the greatest cylinder which can be cut from a given cone.

(Suppose the preceding figure to be made to revolve round the axis of x , and to describe a cone of altitude a and radius of base b ; also a cylinder of altitude x , and radius of base y . Then the volume of the cylinder

$$V = \pi xy^2 = \pi b^2 x \left(1 - \frac{x}{a}\right)^2;$$

$$\frac{dV}{dx} = \pi b^2 \left\{ \left(1 - \frac{x}{a}\right)^2 - 2\frac{x}{a} \left(1 - \frac{x}{a}\right) \right\} = \pi b^2 \left(1 - \frac{x}{a}\right) \left(1 - 3\frac{x}{a}\right) = 0,$$

when $x = a$, or $x = \frac{1}{3}a$.

(i.) When $x = a$, $\frac{d^2V}{dx^2} = 2\pi\frac{b^2}{a}$; and $V = 0$, a minimum;

(ii.) When $x = \frac{1}{3}a$, $\frac{d^2V}{dx^2} = -2\pi\frac{b^2}{a}$; and

$$V = \frac{4}{27}\pi ab^2 = \frac{4}{9} \text{ volume of the cone, a maximum.})$$

(15) Determine the cylinder of greatest curved surface which can be inscribed in a given cone.

(With the same notation as before, the surface $S = 2\pi xy$, and therefore, as in Ex. 13, S is a maximum when

$$x = \frac{1}{2}a, y = \frac{1}{2}b.)$$

- (16) Determine the greatest cylinder which can be cut from a given sphere.

(Here the volume $V = 2\pi xy^2$, where $y^2 = a^2 - x^2$; so that

$$V = 2\pi(a^2x - x^3);$$

and

$$dV/dx = 2\pi(a^2 - 3x^2) = 0,$$

when

$$x^2 = \frac{1}{3}a^2, \quad x = \frac{1}{\sqrt{3}}\sqrt{3}a; \quad y/x = \sqrt{2};$$

and then

$$V = \frac{4}{3}\sqrt{3}\pi a^3 = \frac{1}{3}\sqrt{3} \text{ volume of the sphere.})$$

- (17) Determine the cylinder of greatest curved surface which can be inscribed in a given sphere.

(Here the surface $S = 4\pi xy = 4\pi x\sqrt{(a^2 - x^2)}$.

We can rationalize S , and omit constant factors, and determine the maximum value of this new quantity.

Let

$$u = x^2(a^2 - x^2),$$

then

$$du/dx = 2a^2x - 4x^3 = 0,$$

when

$$x^2 = \frac{1}{2}a^2, \quad x = \frac{1}{\sqrt{2}}\sqrt{2}a;$$

and then

$$d^2u/dx^2 = -4a^2;$$

so that u and therefore S is a maximum when $x = \frac{1}{\sqrt{2}}a\sqrt{2}$, and then $S = 2\pi a^2 = \frac{1}{2}$ surface of the sphere.)

- (18) Prove that to make the whole surface of the cylinder a maximum, the height must be $\frac{1}{\sqrt{2}}\sqrt{2}$ of the diameter of the sphere.

- (19) Prove that the volume of the greatest cone which can be cut from a given sphere is $\frac{8}{27}$ of its volume.

- (20) Prove that the cone of least volume which will contain a given sphere has twice the volume of the sphere, and altitude twice the diameter.

Prove also that the closed cone of least surface which will contain the sphere has the same dimensions.

(Use the semi-vertical angle of the cone as the independent variable.)

- (21) Determine the proportions of a cylinder of given volume, open at one end and closed at the other, when the surface is a minimum.

(The diameter is double the height. This is the problem of the gasholder of given capacity and minimum weight; the gasholder being a cylindrical vessel closed at the top and open at the bottom, where it sinks into water.

Thus a gasholder of 8 million cubic feet capacity should be about 137 feet high and 273 feet in diameter.)

- (22) Determine the proportions of a cylinder of given volume, closed at both ends, in order that the whole surface should be a minimum.

(The diameter equal to the length or height.

This is the problem required in the design of a cylindrical boiler of given volume, and minimum weight.)

- (23) Determine the proportions of a cylindrical tin canister to require minimum material for given volume, supposing the ends doubled down to overlap cylindrically (i.) a given distance, (ii.) a given fraction of the length of the cylinder.

(The diameter equal to (i.) the difference, (ii.) the sum of the lengths of the cylinder and of the ends.)

- (24) Prove that a conical tent of given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base; and that the canvas will then, when laid out flat, form a sector of a circle of angle $\frac{2}{3}\sqrt{3}\pi$ radians, or about 208° .

- (25) Prove that, according to the regulations of the Parcel Post, which require the sum of the length and girth of a parcel not to exceed 6 feet—

- (i.) The greatest sphere allowed is about $17\frac{3}{8}$ inches in diameter, and a little over $1\frac{1}{2}$ cubic feet in volume;

- (ii.) The greatest cube is $14\frac{2}{3}$ inches long, and nearly $1\frac{3}{4}$ cubic feet in volume ;
- (iii.) The greatest rectangular box is 2 feet long and 1 foot square, and 2 cubic feet in volume ;
- (iv.) The greatest parcel of any shape is a cylinder, 2 feet long, and 4 feet in girth, and over $2\frac{1}{2}$ cubic feet in volume. (*Rev. W. A. Whitworth.*)

(26) Determine the speed most economical in fuel to steam against a tide, supposing the resistance to vary as the n^{th} power of the velocity through the water.

(*Solution.*—Let a denote the velocity of the tide, x the velocity of the steamer through the water ; then $x - a$ will be the velocity of the steamer relatively to the bank.

The power required and therefore the coal burnt per hour will vary as the product of the resistance and the speed, that is, as x^{n+1} , and therefore the coal burnt per mile will vary as $x^{n+1}/(x - a)$.

This is a min. when $x/a = 1 + 1/n$, or $(x - a)/a = 1/n$.

Thus if the resistance is taken to vary as the square of the velocity, the speed past the bank should be half the velocity of the current.)

(27) Determine the length of cartridge which will give maximum velocity to a projectile in a gun, supposing the powder instantaneously ignited, and that in expanding its pressure varies inversely as the m^{th} power of its volume (ex. 3, p. 101).

(Denoting by l the length of the bore, we must make

$$\frac{1}{4}\pi d^2 P a \{1 - (a/l)^{m-1}\} / (m - 1)$$

a maximum by variation of a ; so that

$$1 - m(a/l)^{m-1} = 0, \text{ or } a/l = (1/m)^{1/(m-1)}.$$

When $m = 1$, this gives $a/l = 1/e = \cdot 368$).

*72. In finding a maximum or minimum, exceptional cases sometimes occur, where for a certain value of x , not only $\frac{dy}{dx}=0$, but also $\frac{d^2y}{dx^2}=0$, $\frac{d^3y}{dx^3}=0$,

In such cases it is generally simpler to notice that as x increases continuously, dy/dx changes sign from positive to negative as y passes through a maximum value; and dy/dx changes sign from negative to positive as y passes through a minimum value; but if dy/dx does not change sign, y is neither a maximum or minimum.

Sometimes also y has a maximum or minimum value when dy/dx changes sign by passing through the value infinity; but these cases require special investigation, and are conveniently solved by tracing the curve whose equation is $y=fx$.

These cases are represented graphically in fig. 31.

At A , y has a maximum, and at B , a minimum value.

At C , $dy/dx=0$, but does not change sign, so that y is neither a maximum or minimum, and C is called a *point of inflexion* on the curve.

At D , $dy/dx=\infty$, and changes sign from positive to negative, so that y is a maximum, and D is called a *cusp*.

At E , $y=\infty$, and $dy/dx=\infty$, but does not change sign, and y changes sign from $-\infty$ to $+\infty$ on crossing the *asymptote*.

At F , $y=\infty$, and $dy/dx=\infty$, and changes sign from positive to negative, and y has an infinite max. value.

At G , dy/dx is discontinuous, and changes abruptly from a negative value to a positive value, and y is a minimum.

At H , y has a maximum value, but dy/dx does not vanish. (De Morgan, *Diff. and Int. Calculus*, p. 45.)

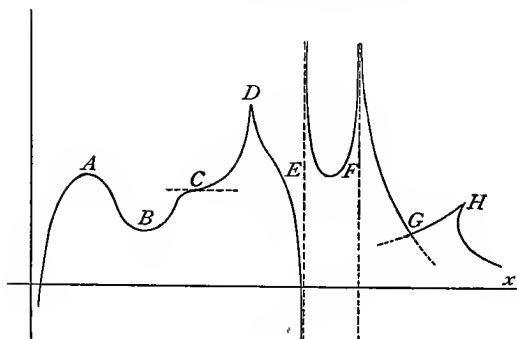


Fig.31.

Supposing the curve to represent the plan of a railway, then we may compare *D* to a station like Cannon Street station; while *G* may represent a turntable, and *H* an engine house on a siding.

Examples.

- (1) If $y = x^5 - 5x^4 + 5x^3$,
 then $\frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2 = 5x^2(x-1)(x-3) = 0$
 when $x = 0$, or 1 , or 3 .

When $x = 0$, $\frac{d^2y}{dx^2} = 0$, and $\frac{dy}{dx}$ does not change sign, so that y is neither a maximum or minimum, as at *C*.

When $x = 1$, $\frac{d^2y}{dx^2} = -10$, and $y = 1$, a maximum; and when $x = 3$, $\frac{d^2y}{dx^2} = 90$, and $y = -27$, a minimum.

- (2) If $\frac{dy}{dx} = (x-1)(x-2)^2(x-3)^3$,
 then $\frac{dy}{dx} = 0$, when $x = 1$, or 2 , or 3 .

When $x = 1$, $\frac{dy}{dx}$ changes from positive to negative, and y is a maximum; when $x = 2$, $\frac{dy}{dx}$ does not change sign, and y is neither a maximum or minimum; similarly when $x = 3$, y is a minimum.

- (3) If $y = x^m(a-x)^n$,
 then $dy/dx = x^{m-1}(a-x)^{n-1}\{ma - (m+n)x\} = 0$,
 when $x=0$, or $ma/(m+n)$, or a .

When $x=0$, dy/dx changes sign from negative to positive, if m is even, and y is then a minimum; but dy/dx does not change sign if m is odd, and then y is neither a maximum or minimum.

Similarly when $x=a$, y is a minimum if n is even; y is neither a maximum or minimum if n is odd.

Therefore the intermediate value $x = ma/(m+n)$ makes y a maximum.

- (4) If $y = a - b(x-c)^{\frac{2}{3}}$,
 then $dy/dx = \infty$, when $x=c$, and changes from positive to negative, so that $y=a$ is a maximum, as at D (fig. 31).

- (5) If $y = (x^{\frac{2}{3}} + 1)(x-7)^2$,
 $dy/dx = \frac{2}{3}x^{-\frac{1}{3}}(x^{\frac{1}{3}} - 1)(x-7)(4x^{\frac{2}{3}} + 4x^{\frac{1}{3}} + 7)$.

When $x=0$, $dy/dx = \infty$, and changes from negative to positive, so that y is a minimum.

When $x=1$, y is a max.; when $x=7$, y is a minimum.

- (6) Discuss the maximums and minimums of y when

- (i.) $y = (x-1)^{\frac{1}{3}}(x-2)^{\frac{2}{3}}(x-3)^{\frac{3}{2}}$;
- (ii.) $dy/dx = (x-1)^{\frac{1}{3}}(x-2)^{\frac{2}{3}}(x-3)^{-3}$;
- (iii.) $dy/dx = (x-a)(x-\beta)^2(x-\gamma)(x-\delta)$;
- (iv.) $dy/dx = (x-a)(x-\beta)^3(x-\gamma)(x-\delta)$; $a < \beta < \gamma < \delta$.

*73. Some maximum and minimum problems, which would otherwise be very complicated, can be solved very simply from the consideration that the function is a maximum or minimum, when it lies between two adjacent values of the function which are equal.

- (1) Suppose it is required to determine the path APB of a ray of light from A to B which shall take the shortest time, supposing the velocity of light to change from v to v' in crossing the curve PP' .

The time from A to B by P is $AP/v + PB/v'$, and by P' is $AP'/v + P'B/v'$ (fig. 32); so that the difference of the time by the two routes is

$$(AP' - AP)/v - (PB - P'B)/v' = Pq/v - Pr/v';$$

when $P'q$, $P'r$ are arcs of circles with centres at A and B .

Then from the consideration that the time in adjacent paths $AP'B$ and APB is the same if a max. or a min. lies between them,

$$Pq/v = Pr/v';$$

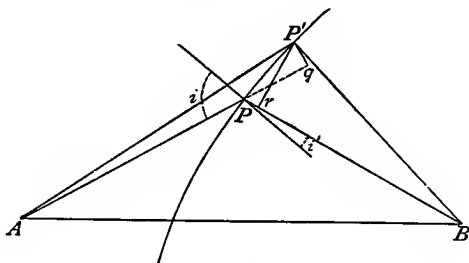


Fig. 32.

$$\frac{v}{v'} = \lim \frac{Pq}{Pr} = \frac{\sin i}{\sin i'},$$

where i , i' are the angles AP , BP make with the normal to the curve PP' at P . In this manner Fermat interpreted Snell's law of the *refraction* of light.

We may suppose PP' to represent a sea-shore, and v the velocity with which a man can row, v' the velocity with which he can walk or ride; and then APB will be his quickest route from A to B .

- (2) To determine the point P , such that the sum of its distances from three given points A, B, C is a minimum; suppose, firstly, that the sum of the distances from two of the given points, B and C , is given; then P is constrained to move on an ellipse of which B and C are the foci; and then the distance AP is a minimum when AP is a normal to this ellipse, and therefore makes equal angles with BP and CP .

Therefore, by symmetry, AP, BP, CP make angles of 120° with each other when $AP + BP + CP$ is a minimum.

- (3) In the same way it may be proved that a triangle DEF inscribed in a triangle ABC has a minimum perimeter when FD, DE make equal angles with $BC: DE, EF$ with $CA:$ and EF, FD with AB ; and then D, E, F are the feet of the perpendiculars drawn from A, B, C on the opposite sides.
- (4) Again, to find the maximum triangle with given base and given vertical angle, since the vertex in this case is constrained to move on the arc of a circle, the area will be a maximum when the altitude is a maximum, and the triangle is then isosceles.
- (5) Some geometrical problems of maximum and minimum can by *projection* (Salmon or Smith's *Conic Sections*) be made to depend on a simpler problem, the solution of which is intuitive.

Thus to find the maximum ellipse which can be inscribed in a triangle, or the minimum ellipse which can be described about a triangle, project the triangle orthogonally into an equilateral triangle; the maximum inscribed ellipse is then the inscribed circle, and the minimum circumscribed ellipse is the circumscribed circle.

- (6) The problem of finding the maximum rectangle which can be inscribed in an ellipse, or the maximum rectangular parallelepiped which can be inscribed in an ellipsoid, is thus by projection reduced to the corresponding problem for a circle and a sphere, the solution of which is a square and a cube.
- (7) Sometimes mechanical considerations are useful; thus of all plane figures with given perimeter, the circle has the greatest area; and of all solids with given surface, the sphere has the greatest volume; as seen in a soap bubble.

The celebrated problem of the shape of the bee cell, for the greatest economy of wax, is seen to be the same problem as the arrangement of the capillary films of an aggregation of regular soap bubbles, which tend to arrange themselves so that the surface is a minimum.

The surface tension being uniform, three faces will meet in an edge at equal angles of 120° ; and the corners or summits will be formed by the meeting of four or eight edges, so that the simplest and most regular elementary cell is a *rhombic dodecahedron*, the figure assumed by plastic spherules like lead bullets in a shrapnel shell, when closely packed in regular order and then compressed into a solid mass by external pressure or shock. (Proc. London Math. Society, vol. xvi. *Cell Structure*, by Mrs. Bryant, D.Sc.)

The corner where four edges of the capillary films meet can be formed by dipping a regular tetrahedron formed on a skeleton frame of soda water wire into the soapy mixture; when four plane films can be made to pass through the edges, and meet at a point in the centre of the tetrahedron, and such a corner is stable.

The other corners, where eight edges meet, are unstable; and can be seen proceeding from the corners to the centre when a skeleton wire cube replaces the tetrahedron; in such cases a small cubelet forms at the centre corner, the edges and faces of which are slightly curved. (Phil. Mag. 1887. *On the Division of Space with Minimum Partitional Area*, by Sir W. Thomson.)

74. *Dynamical Applications of Successive Differentiation.*

Suppose a body M , like a railway train, is moving in the straight line Ox (fig. 1), and that x denotes its distance from O at the time t ; then (§ 10), if u denotes the velocity of the train, $u = dx/dt$.

The *acceleration* is defined to be the *rate of change of velocity, or the growth, per unit of time* (§ 11); so that the acceleration of M is $\frac{du}{dt} = \frac{d^2x}{dt^2}$.

Also, with x for independent variable,

$$\frac{du}{dt} = \frac{du}{dx} \frac{dx}{dt} = \frac{du}{dx} u = \frac{d\frac{1}{2}u^2}{dx}.$$

Suppose, for example, the acceleration is constant and equal to f ; then $\frac{du}{dt} = \frac{d^2x}{dt^2} = f$.

Integrating with respect to t ,

$$u = dx/dt = U + ft \dots\dots\dots(1),$$

if U denotes the velocity when $t=0$; so that U is the arbitrary constant to be added in integration (§ 39).

Integrating again with respect to t ,

$$x = Ut + \frac{1}{2}ft^2 = \frac{1}{2}(U+u)t \dots\dots\dots(2)$$

supposing $x=0$, when $t=0$; so that $\frac{1}{2}V(+v)$ is the *average* velocity, the arithmetic mean of the initial velocity U and the final velocity u , the velocity attained in time $\frac{1}{2}t$.

Again $d\frac{1}{2}u^2/dx=f$,

so that, integrating with respect to x ,

$$\frac{1}{2}u^2=\frac{1}{2}U^2+fx\ldots\ldots\ldots(3);$$

and (1), (2), (3) are the equations of rectilinear motion with constant acceleration f .

For a retardation f is negative.

75. Vertical Motion under Gravity, when unresisted.

Suppose a body N (fig. 1) to be projected vertically upwards from O with velocity V ; and let y denote the height above O and v the upward velocity of N at the time t .

Then $v=dy/dt$; and if the resistance of the air is left out of account,

$$\frac{dv}{dt}=\frac{d^2y}{dt^2}=-g,$$

g denoting the acceleration of *gravity*, due principally to the attraction of the Earth.

Therefore, integrating with respect to t ,

$$v=dy/dt=V-gt,$$

and integrating again

$$y=Vt-\frac{1}{2}gt^2=\frac{1}{2}(V+v)t,$$

also, from (3), $\frac{1}{2}v^2=\frac{1}{2}V^2-gy$.

If the body N rises to the height h , and if T denotes the whole time of going up and coming down again, then $\frac{1}{2}T$ is the time of rising and also of falling, and

$$V=\frac{1}{2}gT, \quad \frac{1}{2}V^2=gh, \quad \text{and} \quad h=\frac{1}{8}gT^2.$$

Thus with a foot and second as units, and $g=32$, then $h=(2T)^2$, or the height attained in feet is the square of twice the number of seconds the body is in the air.

Also $v=g(\frac{1}{2}T-t)$,

and $y=\frac{1}{2}gt(T-t)=\frac{1}{2}gt't'$, if $t'=T-t$;

$$\frac{1}{2}v^2=g(h-y).$$

*76. *Vertical motion of a body in a resisting medium.*

When a body like a sphere is projected vertically in the air or any other resisting medium, in which it is assumed that the resistance varies as the n th power of the velocity; then if the weight of the body is W lb., and if w denotes its *terminal velocity*, that is the velocity with which it descends uniformly when the upward resistance of the medium balances the downward attraction of gravity, the resistance of the air at any other velocity v will be a force of $W(v/w)^n$ pounds, and the retardation due to this resistance will be $g(v/w)^n$.

The terminal velocity is observable in falling rain-drops, hailstones, or meteorites; also in a train or steamer at full speed. In a balloon or parachute we seek to make the terminal velocity as small as possible, but in projectiles we increase the terminal velocity and thereby the ranging power by giving them an elongated form.

When the body is moving *downwards* with velocity v , the resistance of the medium acts upwards, and the equation of motion is

$$\frac{dv}{dt} = g - g\left(\frac{v}{w}\right)^n \dots\dots\dots(1);$$

and when the body is moving *upwards*, with velocity v , both gravity and the resistance act downwards, and the equation of motion is

$$\frac{dv}{dt} = -g - g\left(\frac{v}{w}\right)^n \dots\dots\dots(2).$$

We shall take the resistance to vary as the square of the velocity, so that $n=2$; and now equation (1) may be

written
$$\frac{dt}{dv} = \frac{1}{g} \frac{1}{1 - (v/w)^2},$$

so that
$$t = \frac{1}{g} \int_0^v \frac{dv}{1 - (v/w)^2}$$

$$= \frac{w}{2g} \log \frac{w+v}{w-v}, \text{ or } \frac{w}{g} \tanh^{-1} \frac{v}{w}, \text{ by (y), p. 85.}$$

or $v = dy/dt = w \tanh gt/w$ (3)
 if y denotes the depth of the body below the highest point after falling a time t (seconds).

Therefore, integrating again,

$$y = w \int_0^{gt/w} \tanh \frac{gt}{w} dt = \frac{w^2}{g} \log \cosh \frac{gt}{w} \dots\dots\dots(4).$$

In the ascending motion, suppose the body was at a depth y below the highest point t' seconds before reaching it, and that its velocity was v' ; then $v' = dy/dt'$, and

from (2)
$$\frac{dv'}{dt'} = g \left(1 + \frac{v'^2}{w^2} \right) \dots\dots\dots(5),$$

so that
$$t' = \frac{1}{g} \int_0^{v'} \frac{dv'}{1 + (v'/w)^2} = \frac{w}{g} \tan^{-1} \frac{v'}{w},$$

or $v' = w \tan gt'/w$ (6),

and
$$y = w \int_0^{gt'/w} \tan \frac{gt'}{w} dt' = \frac{w^2}{g} \log \sec \frac{gt'}{w} \dots\dots\dots(7).$$

From these equations (3) to (7) we find

$$\exp gy/w^2 = \sec gt'/w = \cosh gt/w = v'/v,$$

so that $gt'/w = gd \, gt/w$, and $(w/v')^2 - (w/v)^2 = 1$.

If in fig. 13 the circular sector OAP is taken proportional to t' , the time of ascent to a height y , then the hyperbolic sector OAQ will bear the same ratio to t , the time of falling back again; and the velocities v' of projection and v of fall will be represented by AR and AU , while the height y will be proportional to the logarithm of the secant OT . Also at half the times of ascent and descent the velocities will be equal, and represented by At . (Newton, *Principia*, lib. ii., Prop. ix.)

The upward velocity is therefore infinite at an infinite depth below the highest point, but the time of ascent is finite, namely $\frac{1}{2}\pi w/g$.

In the downward motion the velocity gradually grows from zero to w the terminal velocity, which is reached asymptotically at an infinite depth, and in an infinite time.

The terminal velocity of a rifle bullet may be taken as 400 f.s.; and therefore, if fired vertically upwards with velocity 1200 f.s., instead of attaining in the absence of resistance a height of 22500 feet in $37\frac{1}{2}$ seconds, it will ascend only 5750 feet in 15 seconds; returning to the ground again in 23 seconds with velocity 380 f.s.

To attain within one per cent. of the terminal velocity, when v/w is equal to or less than .99, occupies a time

$$t = \frac{w}{2g} \log_e 199 = \frac{w}{2g} \times 5.2933,$$

during which the body will have fallen

$$y = \frac{w^2}{g} \log_e \frac{100}{\sqrt{(199)}} = \frac{\frac{1}{2}w^2}{g} \times 3.9171.$$

Thus if a man falls in a parachute 800 yards in 2 minutes, we may suppose $w=20$; and with $g=32$, we find $t=1.7$, and $y=24.4$, the time in seconds and the distance fallen in feet to attain within one per cent. of the terminal velocity.

In these equations g represents the acceleration of gravity, corrected for buoyancy; so that it may happen, as with a balloon in the air, or a buoyant body rising to the surface in water, that the sign of g must be changed.

In the motion of a steamer through the water against a resistance varying as the square of the velocity, or of a railway train moving against a resistance consisting of a constant part and a part proportional to the square of the

velocity, we must replace g by f , the acceleration with which the steamer or train starts or stops, and take w to represent the full speed. Thus if a steamer of W tons displacement is propelled at a velocity w (feet per second) with a uniform thrust of T tons, $f = gT/W$, and the horsepower at full speed is $2240Tw/550$; and to stop by reversing the engines takes $\frac{1}{4}\pi w/f$ seconds, during which the steamer will have gone $(\frac{1}{2}w^2/f)\log_e 2 = (\frac{1}{2}w^2/f) \times 0.6931$ feet.

**(77) Experimental Determination of the Resistance of the Air.*

The velocity of projectiles and the resistance of the air is now inferred experimentally from the instants of time recorded by a chronograph at which a series of equidistant screens are passed, the record being made by the projectile cutting wires carrying an electric current; the projectile is supposed to travel so fast that it may be taken to fly in a horizontal straight line.

If the shot takes t seconds to go s feet, then the velocity v and the acceleration f are given by (§§ 10, 11, 74)

$$v = \frac{ds}{dt}, f = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v \frac{dv}{ds}.$$

But the chronograph gives t as a function of s , not s as a function of t , so that we must write $v = 1/\frac{dt}{ds}$; while

$$f = v \frac{dv}{ds} = v \frac{d}{ds} \left(1/\frac{dt}{ds} \right) = v \left(-\frac{d^2t}{ds^2} \right) / \left(\frac{dt}{ds} \right)^2 = -\frac{d^2t}{ds^2} v^3.$$

Then if the shot weighs W lb., the resistance of the air is $W \frac{d^2t}{ds^2} v^3$ poundals, or $W \frac{d^2t}{ds^2} v^3/g$ pounds.

Mr. Bashforth in his experiments found that d^2t/ds^2 was a slowly varying quantity; and if we take it as constant,

we assume that the resistance varies as the cube of the velocity; and now if $d^2t/ds^2=c$, and we integrate, supposing V is the initial velocity, where $s=0$ and $t=0$,

$$\frac{1}{v} = \frac{dt}{ds} = \frac{1}{V} + cs, \text{ and } t = \frac{s}{V} + \frac{1}{2}cs^2 = \frac{1}{2}\left(\frac{1}{V} + \frac{1}{v}\right)s.$$

The *average* velocity U over s is given by $U=s/t$, and

$$\frac{1}{U} = \frac{t}{s} = \frac{1}{V} + \frac{1}{2}cs = \frac{1}{2}\left(\frac{1}{V} + \frac{1}{v}\right);$$

so that the average velocity U is the actual velocity at the distance $\frac{1}{2}s$ of the mid point, and is the harmonic mean of the initial velocity V and the final velocity v .

Suppose that three equidistant screens l feet apart are cut at instants of time by chronograph t_1, t_2, t_3 seconds; then, with $d^2t/ds^2=c$, we assume that $t=a+bs+\frac{1}{2}cs^2$.

Measuring s backwards and forwards from the middle screen, we find $a=t_2$, $2bl=t_3-t_1$, $cl^2=t_1-2t_2+t_3$; and at the middle screen $V=1/b=2l/(t_3-t_1)$, the average velocity from the first to the third screen; while the resistance of the air at this velocity is WcV^3/g pounds.

For instance if $W=70, l=150$ and $t_1=2.3439, t_2=2.4325, t_3=2.5221$; then $V=1684$ f.s., and the resistance of the air is 462 pounds. (*The Bashforth Chronograph*, 1890.)

When a projectile, whose terminal velocity is w , is flying horizontally against a resistance of the air varying as the n th power of the velocity, the equations of motion

are
$$\frac{dv}{dt} = \frac{v dv}{ds} = -g\left(\frac{v}{w}\right)^n;$$

so that
$$\frac{gt}{w} = \int_v^V \frac{u^{n-1} dv}{v^n} = \frac{1}{n-1} \left\{ \left(\frac{w}{v}\right)^{n-1} - \left(\frac{w}{V}\right)^{n-1} \right\};$$

$$\frac{gs}{w^2} = \int_v^V \frac{u^{n-2} dv}{v^{n-1}} = \frac{1}{n-2} \left\{ \left(\frac{w}{v}\right)^{n-2} - \left(\frac{w}{V}\right)^{n-2} \right\}.$$

But when $n=1$, $gt/w = \log V/v$, $gs/w^2 = V-v$;
 and when $n=2$, $\frac{gt}{w} = \frac{w}{v} - \frac{w}{V}$, $\frac{gs}{w^2} = \log \frac{V}{v}$.

78. Equations of Motion in a Plane.

If x, y are the coordinates at the time t of a point P moving in a plane (fig. 1), then (§ 10), $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the component velocities of P parallel to Ox and Oy .

Similarly, $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ are the component *accelerations* parallel to Ox and Oy ; since the acceleration in any direction is defined to be the *rate of change of velocity in that direction, reckoned by the growth per unit of time.*

79. Motion of a Projectile when Unresisted.

Suppose, for instance, that a body P is projected from O (fig. 33) with velocity V at an angle α to the horizon, and that the resistance of the air is left out of account.

Then, if the axis Ox is horizontal, and the axis Oy drawn vertically upwards, the equations of motion are

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -g.$$

Integrating with respect to t :

$$\frac{dx}{dt} = \text{a const.} = V \cos \alpha, \quad \frac{dy}{dt} = \text{a const.} - gt = V \sin \alpha - gt;$$

and integrating again, supposing $t=0$ at O ,

$$x = Vt \cos \alpha, \quad y = Vt \sin \alpha - \frac{1}{2}gt^2.$$

Therefore $t = x/(V \cos \alpha)$, and substituting this value of

$$t \text{ in } y, \quad y = x \tan \alpha - \frac{gx^2}{2V^2 \cos^2 \alpha},$$

the equation of the trajectory; which will be found to be a parabola.

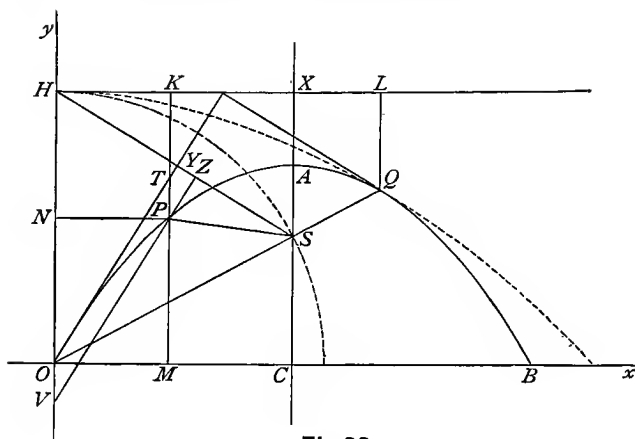


Fig.33.

For treating the equation as a quadratic in x ,

$$\left(x - \frac{V^2}{g} \sin \alpha \cos \alpha\right)^2 = -\frac{2V^2}{g} \cos^2 \alpha \left(y - \frac{V^2}{2g} \sin^2 \alpha\right);$$

and comparing this with the equation

$$(x-h)^2 = -p(y-k),$$

which is the equation of a parabola as in fig. 33, of which the vertex is at (h, k) , the latus rectum is p , and the concavity downwards; then

$$h = (V^2 \sin \alpha \cos \alpha)/g, \quad k = \frac{1}{2}(V^2 \sin^2 \alpha)/g, \quad p = 2(V^2 \cos^2 \alpha)/g.$$

If HK is the directrix of the parabola, then

$$OH = k + \frac{1}{4}p = \frac{1}{2}V^2/g.$$

The height $\frac{1}{2}V^2/g$ a body must fall to acquire the velocity V was called the *impetus* of the velocity v ; so that the impetus at O is OH .

Again
$$\frac{dx}{dt} \frac{d^2x}{dt^2} + \frac{dy}{dt} \frac{d^2y}{dt^2} = -g \frac{dy}{dt};$$

and integrating, v denoting the velocity at P ,

$$\frac{1}{2}v^2 = \frac{1}{2} \left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} \right) = \frac{1}{2}V^2 - gy.$$

The velocity v at any point P is therefore the velocity which would be acquired in falling freely from the level of the directrix; for $\frac{1}{2}v^2 = g \cdot OH - g \cdot MP = g \cdot PK$; or the impetus of the velocity at any point P is PK , the depth below the directrix.

Produce MP to meet the tangent at O in T ; then since $OM = x = Vt \cos \alpha$, therefore $OT = Vt$; and since $MP = y = Vt \sin \alpha - \frac{1}{2}gt^2$, and $MT = Vt \sin \alpha$, then $TP = \frac{1}{2}gt^2$.

Thus the parabola OP may be supposed described by a body which is carried in t seconds from O to T by the original velocity of projection V , without being influenced by gravity or resistance (the *motus violentus* of ancient writers), and which afterwards falls from T to P in t seconds under gravity without resistance (the *motus naturalis*), the combination of these two motions being called the *motus mixtus*; this is the method of Galileo employed in Elementary Dynamics.

Then with $OT = Vt$, $TP = \frac{1}{2}gt^2$, the elimination of t gives

$$OT^2/TP = V^2t^2/\frac{1}{2}gt^2 = 2V^2/g = 4OH,$$

so that $OT^2 = 4OH \cdot TP$, or $PV^2 = 4HO \cdot OV$,

(fig. 33), whence it is inferred, by a theorem of Geometrical Conics, that the trajectory OP is a parabola.

But we can easily show that the curve OP satisfies the definition of a parabola as "the locus of a point which moves so that its distance from a fixed point is equal to its distance from a fixed straight line" by making the angle TOS equal to the angle TOH , and OS equal to OH ;

now if OT and the parallel VP meet SH in Y and Z ;
then since $PV^2 = 4HO \cdot OV$,

therefore, by similar triangles,

$$\begin{aligned} PN^2 &= 4HY \cdot YZ = (HY + YZ)^2 - (HY - YZ)^2 \\ &= HZ^2 - SZ^2 = HP^2 - SP^2, \end{aligned}$$

so that $SP^2 = HP^2 - PN^2 = PK^2$, or $SP = PK$,

and P therefore describes a parabola, according to this last elementary definition.

A jet of water issuing from O in the direction OT with velocity V (or a stream of bullets from a Maxim gun), will form a continuous parabola, and since the horizontal component of the velocity is constant, equidistant vertical ordinates will cut off equal volumes of water, or the line density of the jet is proportional to $\cos \psi$; so that the height of the centre of gravity of the jet is the average height of the ordinates or $\frac{2}{3}$ the height of the vertex A , since the area OAB is $\frac{2}{3}$ of the circumscribing rectangle.

The jet if frozen will stand as an arch, even when cracked across by normal planes; and when inverted will hang as a catenary, without change of form if flexible.

In a suspension bridge the weight may be supposed concentrated in a uniform roadway, and the chains will then assume the parabolic curve.

80. To find the range OB , and the time of flight on the horizontal plane through O , put $y = 0$; then

$$0 = x \tan \alpha - gx^2/2V^2 \cos^2 \alpha,$$

$$0 = Vt \sin \alpha - \frac{1}{2}gt^2.$$

Therefore, if the range is denoted by R ,

$$R = 2V^2 \sin \alpha \cos \alpha / g = V^2 \sin 2\alpha / g,$$

so that the elevation α required to attain a range R with velocity V is given by $\alpha = \frac{1}{2} \sin^{-1}(gR/V^2)$; and if the time of flight is denoted by T , then $T = 2(V \sin \alpha)/g$.

With a given velocity of projection V , the range R on a horizontal plane is a maximum when $\sin 2\alpha = 1$, or $\alpha = \frac{1}{4}\pi$.

Produce OS to meet the parabolic trajectory again in Q , and draw QL perpendicular to the directrix HK ; then

$$OQ = OS + SQ = OH + QL,$$

so that the trajectory touches at Q a fixed parabola HQ , with focus at O and vertex at H ; the tangents at O and Q are at right angles, so that the angles of ascent at O and descent at Q are complementary.

The parabola HQ is called the *envelope* of the trajectories like OPQ , described with the same velocity of projection V , but varying elevation α .

The maximum range on a line through O , like OQ , will be obtained when OQ is a focal chord, and then the direction of projection bisects the angle HOQ .

The interior of the surface formed by the revolution of the parabola HQ round the vertical line OH will be the whole space covered from the point O with the given velocity of projection V ; and the section of this surface by any plane will be the area covered on that plane.

Thus the area covered by a gun on an inclined plane is the elliptic section of this surface made by the plane, and the gun will be at a focus of the ellipse if it lies in the plane; again the area covered by a fire engine on a wall at a given distance is a parabola, the section of the surface made by the vertical plane of the wall.

For other examples on the subject of parabolic motion, the reader is referred to treatises on Dynamics.

81. *Dynamical Equations with Polar Coordinates.*

Changing to polar coordinates r and θ , then (fig. 34)

$$x = r \cos \theta, \text{ and } y = r \sin \theta.$$

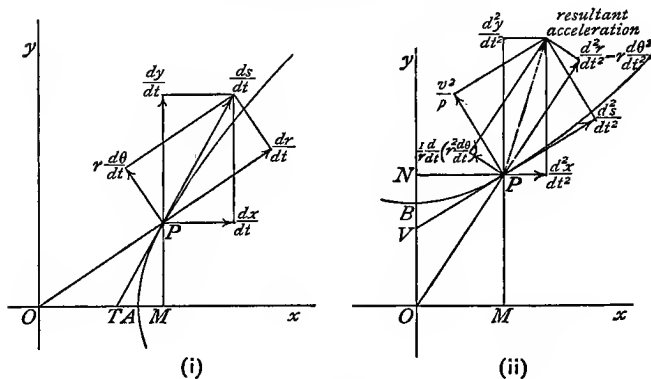


Fig. 34

Differentiating with respect to t ,

$$\frac{dx}{dt} = \frac{dr}{dt} \cos \theta - r \sin \theta \frac{d\theta}{dt}, \quad \frac{dy}{dt} = \frac{dr}{dt} \sin \theta + r \cos \theta \frac{d\theta}{dt}.$$

Therefore the component velocity in the direction OP , called the *radial velocity*, as before (§ 23),

$$= \frac{dx}{dt} \cos \theta + \frac{dy}{dt} \sin \theta = \frac{dr}{dt};$$

and the component velocity perpendicular to OP , called the *transversal velocity*,

$$= -\frac{dx}{dt} \sin \theta + \frac{dy}{dt} \cos \theta = r \frac{d\theta}{dt}.$$

Differentiating again,

$$\frac{d^2x}{dt^2} = \left(\frac{d^2r}{dt^2} - r \frac{d\theta^2}{dt^2} \right) \cos \theta - \left(2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right) \sin \theta,$$

$$\frac{d^2y}{dt^2} = \left(\frac{d^2r}{dt^2} - r \frac{d\theta^2}{dt^2} \right) \sin \theta + \left(2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right) \cos \theta.$$

Therefore the component *radial acceleration*

$$\frac{d^2x}{dt^2} \cos \theta + \frac{d^2y}{dt^2} \sin \theta = \frac{d^2r}{dt^2} - r \frac{d\theta^2}{dt^2},$$

and the component *transversal acceleration*

$$-\frac{d^2x}{dt^2} \sin \theta + \frac{d^2y}{dt^2} \cos \theta = 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2},$$

which is usually written in the form $\frac{1}{r} \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right)$, to which it is equivalent, as may be verified by differentiation.

82. Motion in a Circle.

Suppose, for instance, that P describes a circle round O as centre; then r is constant and $dr/dt=0$; so that the radial acceleration is $-rd\theta^2/dt^2$ and the transversal or tangential acceleration is $rd^2\theta/dt^2$.

Also, since $s=r\theta$, if v denotes the velocity in the circle,

$$v=ds/dt=r d\theta/dt=n r;$$

where $n=d\theta/dt$, called the *angular velocity* round O ;

so that the tangential acceleration is $\frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$ or $v \frac{dv}{ds}$,

the same as for rectilinear motion; and the central or normal acceleration in the direction PO is

$$r(d\theta/dt)^2=v^2/r=n^2r=nv.$$

To realize this circular motion practically, suppose a plummet, suspended from a fixed point by a fine thread of length l , to be projected with velocity v so as to describe a horizontal circle of radius $r=l \sin \alpha$ under gravity.

Denoting by Q the tension of the thread, then

$$v^2/r : g :: Q \sin \alpha : Q \cos \alpha, \text{ or } v^2/r = g \tan \alpha.$$

Now if T denotes the *period*, or time of revolution in the circle, $T=2\pi r/v$; therefore $g \tan \alpha = v^2/r = 4\pi^2 r/T^2$,

or $T=2\pi \sqrt{(r \cot \alpha/g)}=2\pi \sqrt{(l \cos \alpha/g)}$,

and therefore, in the limit when $\alpha=0$, $T=2\pi \sqrt{(l/g)}$.

*83. *Motion in a Field of Force.*

Suppose a body, considered as a particle, to move in a *field of force* in which the radial and transversal accelerations due to the field of force are represented by R and T respectively; then the equations of motion are

$$\frac{d^2r}{dt^2} - r \frac{d\theta^2}{dt^2} = R \dots \dots \dots (1),$$

$$\frac{1}{r} \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) = T \dots \dots \dots (2).$$

Now $r^2 d\theta/dt$ is generally denoted by h , and then $h = 2dA/dt$ (§ 56), twice the rate the sectorial area A is swept out by the vector OP , revolving about O .

Change the independent variable in the equations of motion from t to θ , and denote r by $1/u$ (§ 23).

Then since $r^2 d\theta/dt = h$, therefore $d\theta/dt = hu^2$; and equation (2) becomes

$$\frac{dh}{dt} = \frac{T}{u}, \text{ or } \frac{dh}{d\theta} = \frac{T}{hu^3}, \text{ or } \frac{d\frac{1}{2}h^2}{d\theta} = \frac{T}{u^3}.$$

Again
$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = -\frac{1}{u^2} \frac{du}{d\theta} hu^2 = -\frac{du}{d\theta} h;$$

so that
$$\frac{d^2r}{dt^2} = -\frac{d^2u}{d\theta^2} \frac{d\theta}{dt} h - \frac{du}{d\theta} \frac{dh}{dt} = -\frac{d^2u}{d\theta^2} h^2 u^2 - \frac{du}{d\theta} \frac{T}{u}.$$

Then equation (1) becomes, since $r(d\theta/dt)^2 = h^2 u^3$,

$$-\frac{d^2u}{d\theta^2} h^2 u^2 - \frac{du}{d\theta} \frac{T}{u} - h^2 u^3 = R,$$

or
$$\frac{d^2u}{d\theta^2} + u = -\frac{R}{h^2 u^2} - \frac{T}{h^2 u^3} \frac{du}{d\theta} \dots \dots \dots (3),$$

the *differential equation of the orbit of P.*

*84. *Central Orbits.*

For a central field of force in which the attraction is always directed to the origin O , $T=0$, and h is constant.

Denoting the central acceleration due to the attraction towards O by P , so that $P = -R$, then

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} \dots \dots \dots (4),$$

or
$$P = h^2u^2 \left(\frac{d^2u}{d\theta^2} + u \right),$$

whence the required value of P is found when the equation of the orbit is given.

Examples.

- (1) The polar equation of a conic section with a focus at the origin being l/r or $lu = 1 + e \cos \theta$, then

$$l \left(\frac{d^2u}{d\theta^2} + u \right) = 1,$$

and $P = h^2u^2/l = n^2a^3r^{-2}$, which varies as u^2 or r^{-2} ; a denoting the mean distance, and n the *mean motion* or mean angular velocity round O ; and therefore $2\pi/n$ the *period* of revolution.

Thus Newton's Law of Gravitation, by which the Sun attracts the planets with intensity inversely proportional to the square of the distance, is deduced from Kepler's Law, that the planets describe ellipses of which the Sun occupies a focus.

- (2) Prove that, for a conic section with the centre at the origin, P varies as r .
- (3) Prove that P varies as u^3 in the curves (*Cotes's Spirals*) (i.) $au = \cosh n\theta$, (ii.) $au = \exp n\theta$ (the equiangular spiral, p. 53), (iii.) $au = \sinh n\theta$, (iv.) $au = n\theta$ (the *reciprocal spiral*), (v.) $au = \sin n\theta$, or $\cos n\theta$, or $\cos n(\theta - a)$.

(4) Prove that P varies as r^{-2n-3} in $r^n = a^n \cos n\theta$.

(5) (i.) P varies as u^5 in $au = \tanh \frac{1}{2}\sqrt{2\theta}$ or $\coth \frac{1}{2}\sqrt{2\theta}$;

(ii.) P varies as u^4 in $au = \frac{\cosh \theta - 2}{\cosh \theta + 1}$ or $\frac{\cosh \theta + 2}{\cosh \theta - 1}$;

(iii.) P varies as u^7 in $a^2u^2 = \frac{\cosh 2\theta - 1}{\cosh 2\theta + 2}$ or $\frac{\cosh 2\theta + 1}{\cosh 2\theta - 2}$.

(In these curves $au = 1$ or $r = a$ when $\theta = \infty$; so that after an infinite number of revolutions round the origin, the curves approach to coincidence with the circle $r = a$, which is therefore called an *asymptotic circle*.)

A body describing this asymptotic circle freely under the central attractions of (i), (ii), (iii), would be unstable, and would be found ultimately describing one of the corresponding curves.)

(6) Prove that the radial and transversal P and T in the orbit (r, θ) become changed to $pP + p(q^2 - 1)h^2u^3$ and pqT in the orbit $(pr, q\theta)$.

(7) Prove that the orbit $au = \text{gd } m\theta$ or $\cos au \cosh m\theta = 1$ can be described with $P = h^2u^3$, $T = mh^2u^3 \sin au$; and the orbit $m\theta = \text{gd } au$ or $\cosh au \cos m\theta = 1$ with $P = h^2u^3$, $T = -mh^2u^3 \sinh au$; and that dr/dt is constant.

The inverse problem of the determination of the orbit when the central force is given as a function of u is more complicated; we multiply equation (4) by $du/d\theta$, and integrate both sides with respect to θ ; when

$$\frac{1}{2} \left(\frac{du}{d\theta} \right)^2 + \frac{1}{2} u^2 = \int \frac{P}{h^2 u^2} du + C,$$

whence θ is found as a function of u by transposition, and a single *quadrature* or integration.

85. *Interchange of the Independent and Dependent Variable.*

In §§ 77 and 83, the practical need in Dynamics of expressing differential coefficients, such as dx/dt , d^2x/dt^2 , in terms of differential coefficients of other variables, independent and dependent, has been illustrated; and the general operation required is called *change of the variable*, which we shall now investigate in its generality, and illustrate still further.

In § 77, we found $\frac{d^2s}{dt^2} = -\frac{d^2t}{ds^2} / \left(\frac{dt}{ds}\right)^3$, thus inverting the independent and dependent variable; and with variables x and y , the general theory is established as follows:—

$$\begin{aligned}\frac{dy}{dx} &= 1 / \frac{dx}{dy} \quad (\S 24), \\ \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(1 / \frac{dx}{dy} \right) = \frac{d}{dy} \left(1 / \frac{dx}{dy} \right) \frac{dy}{dx} = -\frac{d^2x}{dy^2} / \left(\frac{dx}{dy} \right)^3, \\ \frac{d^3y}{dx^3} &= - \left\{ \frac{dx}{dy} \frac{d^3x}{dy^3} - 3 \left(\frac{d^2x}{dy^2} \right) \right\} / \left(\frac{dx}{dy} \right)^6,\end{aligned}$$

and so on; we are now said to have *interchanged the independent and dependent variable*.

Write t for $\frac{dy}{dx}$, a for $\frac{1}{2!} \frac{d^2y}{dx^2}$, b for $\frac{1}{3!} \frac{d^3y}{dx^3}$, ...; and use the Greek letters $\alpha, \beta, \gamma, \dots$ for the corresponding derivatives in which x and y are interchanged; then, by Taylor's Theorem, proved subsequently in § 106, if h and k denote simultaneous small finite increments in x and y ,

$$\begin{aligned}k &= th + ah^2 + bh^3 + ch^4 + \dots, \\ h &= \tau k + ak^2 + \beta k^3 + \gamma k^4 + \dots;\end{aligned}$$

so that the expression of t, a, b, c, \dots in terms of $\tau, \alpha, \beta, \gamma, \dots$, or the interchange of independent and dependent variables, is algebraically the same problem as the *rever-*

sion of series, by which the expansion of k in powers of h is changed into the expansion of h in powers of k .

By substituting for h in powers of k from the second relation in the first relation, we obtain the identity

$$k = t(\tau k + \alpha k^2 + \beta k^3 + \dots) \\ + a(\tau k + \alpha k^2 + \dots)^2 \\ + b(\tau k + \dots)^3 + \dots$$

whence

$$t\tau = 1, \quad t = 1/\tau;$$

$$t\alpha + \alpha\tau^2 = 0, \quad \alpha = -\alpha/\tau^3;$$

$$t\beta + 2\alpha\tau\alpha + b\tau^3 = 0, \quad b = -(\beta\tau - 2\alpha^2)/\tau^5;$$

and, similarly, $c = -(\gamma\tau^2 - 5\alpha\beta\tau + 5\alpha^3)/\tau^7$; ..., as before; and the reversion of series changes

$$h = \tau k + \alpha k^2 + \beta k^3 + \gamma k^4 + \dots$$

into

$$k = \tau^{-1}h - \alpha\tau^{-3}h^2 - (\beta\tau - 2\alpha^2)\tau^{-5}h^3 - (\gamma\tau^2 - 5\alpha\beta\tau + 5\alpha^3)\tau^{-7}h^4 \dots,$$

the next term being

$$-(\delta\tau^3 - 6\alpha\gamma\tau^2 - 3\beta^2\tau^2 + 21\alpha^2\beta\tau - 14\alpha^4)\tau^{-9}h^5.$$

* 86. Reciprocants.

Professor Sylvester has given the name of *Reciprocants* to those functions of the successive derivatives of y with respect to x , which preserve their form unaltered, except for t or dy/dx as a factor, when the independent and dependent variables x and y are interchanged; and he calls the Reciprocants *mixed* or *pure*, according as they do or do not involve t or dy/dx . (*American Journal of Mathematics*, vols. viii. and ix.)

Thus $a, 4ac - 5b^2, a^2d - 3abc - 2b^3, \dots$

are *pure* reciprocants; while

$$(1+t^2)b - 2a^2t, bt - a^2, 2ct - 5ab, \dots$$

are *mixed* reciprocants, as the student may verify; for

instance $4ac - 5b^2 = (4a\gamma - 5\beta^2)/\tau^8;$

$$bt - a^2 = -(\beta\tau - \alpha^2)/\tau^6.$$

When the constants are *eliminated* from a rational algebraical relation $F(x, y) = 0$ between x and y , by means of differentiation, the resulting *eliminant* is a *reciprocant*, as it is immaterial which of the two variables x and y we take for independent variables.

Thus if we eliminate the constants in the general equation of the circle

$$x^2 + y^2 + 2Ax + 2By + C = 0,$$

then
$$x + y \frac{dy}{dx} + A + B \frac{dy}{dx} = 0,$$

$$1 + \frac{dy^2}{dx^2} + (y + B) \frac{d^2y}{dx^2} = 0,$$

$$3 \frac{dy}{dx} \frac{d^2y}{dx^2} + (y + B) \frac{d^3y}{dx^3} = 0;$$

and thence we obtain the reciprocant

$$\left(1 + \frac{dy^2}{dx^2}\right) \frac{d^3y}{dx^3} - 3 \frac{dy}{dx} \left(\frac{d^2y}{dx^2}\right)^2 = 0, \text{ or } (1 + t^2)b - 2a^2t = 0.$$

If we eliminate the constants in the equation of the hyperbola

$$xy + Ax + By + C = 0,$$

we obtain the reciprocant (the *Schwartzian*),

$$2 \frac{dy}{dx} \frac{d^3y}{dx^3} - 3 \left(\frac{d^2y}{dx^2}\right)^2 = 0, \text{ or } bt - a^2 = 0.$$

Similarly it may be shown that the reciprocant $\alpha = 0$ represents straight lines; and $4ac - 5b^2 = 0$ parabolas, whose general equation is

$$(Ax + By)^2 + 2Gx + 2Fy + C = 0;$$

while the elimination of the constants in the general equation of a conic leads to the reciprocant (the *Mongian*)

$$a^2d - 3abc + 2b^3 = 0,$$

by means of the result of ex. 9, p. 139, and the consequent relation $\frac{d^3}{dx^3} \left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}} = 0$; reducing to $\frac{d^2}{dx^2} \left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}} = 0$, for parabolas, as before.

87. *Change of the Independent Variable.*

Next to express the derivatives of y with respect to x in terms of the derivatives of x and y with respect to a new independent variable t , we have

$$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} \quad (\S 11),$$

$$\frac{d^2y}{dx^2} = \left(\frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{d^2x}{dt^2} \frac{dy}{dt} \right) \bigg/ \left(\frac{dx}{dt} \right)^3,$$

$$\frac{d^3y}{dx^3} = \left\{ \left(\frac{dx}{dt} \frac{d^3y}{dt^3} - \frac{d^3x}{dt^3} \frac{dy}{dt} \right) \frac{dx}{dt} - 3 \left(\frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{d^2x}{dt^2} \frac{dy}{dt} \right) \frac{d^2x}{dt^2} \right\} \bigg/ \left(\frac{dx}{dt} \right)^5,$$

and so on; and now we are said to have *changed the independent variable from x to t* .

As in § 85, we have

$$k = h \frac{dy}{dx} + \frac{h^2}{2!} \frac{d^2y}{dx^2} + \dots + \frac{h^n}{n!} \frac{d^ny}{dx^n} + \dots,$$

or
$$k = \epsilon \frac{dy}{dt} + \frac{\epsilon^2}{2!} \frac{d^2y}{dt^2} + \dots + \frac{\epsilon^s}{s!} \frac{d^sy}{dt^s} + \dots;$$

where ϵ denotes the increment in t corresponding to that of h in x and k in y ; so that, if $t = \phi x$, then

$$\frac{1}{n!} \frac{d^ny}{dx^n} = \sum_{s=1}^{s=n} C_{(n,s)} \frac{1}{s!} \frac{d^sy}{dt^s},$$

where $C_{(n,s)}$ is the coefficient of h^n in the expansion of ϵ^s or $\{\phi(x+h) - \phi x\}^s$ in ascending powers of h .

In this way we can find the n^{th} derivatives of $\exp x^2$, $\exp x^3$, $\exp \sqrt{x}$, and generally of $\exp x^p$.

An important change of the independent variable in *differential equations* is given by putting $x = e^t$; then

$$x \frac{dy}{dx} = x \frac{dy}{dt} \bigg/ \frac{dx}{dt} = \frac{dy}{dt},$$

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} = \left(\frac{d}{dt} - 1 \right) \frac{dy}{dt},$$

$$x^3 \frac{d^3 y}{dx^3} = \left(\frac{d}{dt} - 2 \right) x^2 \frac{d^2 y}{dx^2} = \left(\frac{d}{dt} - 2 \right) \left(\frac{d}{dt} - 1 \right) \frac{dy}{dt},$$

and generally, by induction,

$$\begin{aligned} x^n \frac{d^n y}{dx^n} &= \left\{ \frac{d}{dt} - (n-1) \right\} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} \\ &= \left\{ \frac{d}{dt} - (n-1) \right\} \dots \left(\frac{d}{dt} - 2 \right) \left(\frac{d}{dt} - 1 \right) \frac{dy}{dt}. \end{aligned}$$

We must distinguish between $x^n \frac{d^n y}{dx^n}$, the product of x^n and $\frac{d^n y}{dx^n}$, and $\left(x \frac{d}{dx} \right)^n y$, obtained by operating n times by $x \frac{d}{dx}$ on y ; for $\left(x \frac{d}{dx} \right)^n y = \frac{d^n y}{dt^n}$, if $x = e^t$.

*88. *Application to Differential Equations.*

The theory of change of the variable is of great practical use in the solution of differential equations.

The general linear differential equation of the second order may be written

$$y'' + 2Py' + Qy = 0, \text{ or } R, \dots \dots \dots (A),$$

where P, Q, R are given functions of x .

By putting $y = uv$, then

$$u'' + 2\left(\frac{v'}{v} + P\right)u' + \left(\frac{v''}{v} + 2P\frac{v'}{v} + Q\right)u = \frac{R}{v};$$

and the coefficient of u' may be removed by making $v'/v = -P$, or $\log v = -\int P dx$, $v = \exp \int (-P dx)$; and now

$$v''/v + 2Pv'/v + Q = Q - P^2 - P',$$

a quantity, denoted by I , and called the *differential invariant*; so that, in its canonical form,

$$u'' + Iu = R \exp \int P dx = S, \text{ suppose } \dots \dots \dots (B).$$

With $R=0$, and u_1, u_2 denoting two particular solutions of the differential equation $u'' + Iu = 0$, so that $u = Au_1 + Bu_2$ is the most general solution; then

$$u_1'' + Iu_1 = 0, \quad u_2'' + Iu_2 = 0,$$

so that, eliminating I , $u_1'' u_2 - u_1 u_2'' = 0$,

and integrating, $u_1' u_2 - u_1 u_2' = C$, a constant.

Denoting by s the quotient u_1/u_2 , then $s''/s' = -2u_2'/u_2$;

$$\text{and} \quad \frac{s'''}{s'} - \frac{3}{2} \left(\frac{s''}{s'} \right)^2 = 2I \dots \dots \dots (C),$$

a *non-linear* differential equation of the *third* order, in which the left hand side is the *Schwartzian derivative*, (s, x) ; and of which the most general solution is thence

$$s = (au_1 + bu_2)/(Au_1 + Bu_2);$$

since $\{(as+b)/(As+B), x\} = (s, x)$, by ex. 16, p. 140.

Knowing any solution s of this equation (C), then from $u_2'/u_2 = -\frac{1}{2}s''/s'$, we obtain, by integration,

$$\log u_2 = \text{constant} - \frac{1}{2} \log s', \text{ or } u_2 = C^{\frac{1}{2}} s'^{-\frac{1}{2}};$$

and then $u_1 = C^{\frac{1}{2}} s s'^{-\frac{1}{2}}$; so that equation (C) may be written in the same canonical form as (B),

$$\frac{d^2}{dx^2} (s'^{-\frac{1}{2}}) + I s'^{-\frac{1}{2}} = 0.$$

Again, denoting the product $u_1 u_2$ by z , then

$$z' = u_1' u_2 + u_1 u_2',$$

$$z'' = u_1'' u_2 + 2u_1' u_2' + u_1 u_2'' = 2(u_1' u_2' - Iz),$$

$$z''' = 2(u_1'' u_2' + u_1' u_2'') - Iz' - I'z$$

$$= -2I(u_1 u_2' + u_1' u_2) - 2(Iz' + I'z) = -4Iz' - 2I'z,$$

$$\text{or} \quad z''' + 4Iz' + 2I'z = 0 \dots \dots \dots (D),$$

a linear differential equation of the third order, also satisfied by u_1^2 and u_2^2 , and therefore generally by

$$z = au_1^2 + 2bu_1 u_2 + cu_2^2.$$

A first integral of this equation is

$$zz'' - \frac{1}{2} z'^2 + 2Iz^2 + \frac{1}{2} C^2 = 0,$$

a non-linear equation of the second order, one solution of which is

$$z = u_1 u_2 = Cs/s'.$$

Supposing a solution z of this differential equation (D) is known; then since

$$u_1' u_2 - u_1 u_2' = C,$$

therefore
$$\frac{u_1'}{u_1} - \frac{u_2'}{u_2} = \frac{C}{z};$$

and integrating, $\log u_1/u_2 = \log s = \int C dx/z;$

or $u_1/u_2 = s = \exp \int C dx/z;$ while $u_1 u_2 = z;$

so that
$$u_1 = \sqrt{(zs)} = \sqrt{z} \exp \frac{1}{2} \int (C dx/z),$$
$$u_2 = \sqrt{(z/s)} = \sqrt{z} \exp \frac{1}{2} \int (-C dx/z).$$

The solution of the more general equation (B) can now be found, and thence the solution of (A); for if

$$u'' + Iu = S, \text{ while } u_1'' + Iu_1 = 0, u_2'' + Iu_2 = 0;$$

then $u''u_1 - uu_1'' = Su_1, u''u_2 - uu_2'' = Su_2;$

and integrating

$$u'u_1 - uu_1' = \int Su_1 dx, u'u_2 - uu_2' = \int Su_2 dx;$$

whence $u = Cu_1 \int Su_2 dx - Cu_2 \int Su_1 dx;$

to which the *complementary function* $Au_1 + Bu_2$ may be added, to obtain the most general solution.

For instance, the solution of $y'' \pm n^2 y = fx$ is

$$u = A_{\cosh}^{\cos} nx + B_{\sinh}^{\sin} nx + \frac{1}{n} \int_{\sinh}^{\sin} n(x - \xi) f(\xi) d\xi.$$

The linear differential equation of the second order in which $I = kx^m$ is sometimes called *Riccati's equation*; being deduced from Riccati's form (ex. 38, p. 82),

$$t \frac{dy}{dt} - ay + by^2 = ct^p,$$

first by changing the independent variable from t to x , where $t^a = x$, and by changing the dependent variable from y to v , where $y = xv$; when the equation becomes

$$\frac{dv}{dx} + \frac{b}{a}v^2 = \frac{c}{a}x^{p/a-2};$$

and secondly by changing the dependent variable to u ,
where $bv/a = d \log u/dx$,

whence
$$\frac{b}{a} \frac{dv}{dx} = \frac{1}{u} \frac{d^2u}{dx^2} - \left(\frac{1}{u} \frac{du}{dx} \right)^2 = \frac{1}{u} \frac{d^2u}{dx^2} - \frac{b^2v^2}{a^2},$$

so that
$$\frac{1}{u} \frac{d^2u}{dx^2} = \frac{b}{a} \left(\frac{dv}{dx} + \frac{b}{a}v^2 \right) = \frac{bc}{a^2} x^{p/a-2};$$

and therefore $I = -bcx^{p/a-2}/a^2$, of the form kx^m .

This equation is again reduced, by changing the independent variable to $r = x^{1/p}$, to

$$\frac{d^2u}{dr^2} + \left(1 - \frac{2a}{p}\right) \frac{du}{rdr} - \frac{4bc}{p^2}u = 0;$$

and on changing the dependent variable from u to w ,
where $u = wr^{a/p-1}$, to the form employed by Laplace,

$$\frac{1}{w} \frac{d^2w}{dr^2} = \frac{n(n+1)}{r^2} \pm q^2,$$

where n replaces $a/p - \frac{1}{2}$, and $\pm q^2$ is written for $\pm 4bc/p^2$;
and when n is an integer, this equation has a solution
which can be expressed in finite terms.

As an exercise in differentiation the student may verify
that this solution may then be written

$$w = r^{n+1} \left(\frac{1}{r} \frac{d}{dr} \right)^{n+1} (A_{\cos}^{\cosh} qr + B_{\sin}^{\sinh} qr),$$

or
$$w = qr^{n+1} \left(\frac{1}{r} \frac{d}{dr} \right)^n \left(\frac{A_{-\sin}^{\sinh} qr + B_{\cos}^{\cosh} qr}{r} \right);$$

or
$$w = \text{coefficient of } (2h)^{n+1}/(n+1)!$$

in the expansion of

$$A_{\cos}^{\cosh} q\sqrt{(r^2+hr)} + B_{\sin}^{\sinh} q\sqrt{(r^2+hr)};$$

and illustrate especially the cases of $n=1$, and $n=2$.

Putting $u = vr^{a/p}$, or $w = vr^{\frac{1}{2}}$, will give the form employed by Bessel, with $m = a/p = n + \frac{1}{2}$,

$$r^2 \frac{d^2 v}{dr^2} + r \frac{dv}{dr} + (q^2 r^2 - m^2)v = 0.$$

For instance,

$$r^2 \frac{d^2 v}{dr^2} + r \frac{dv}{dr} + (q^2 r^2 - \frac{9}{4})v = 0,$$

where $m = \frac{3}{2}$, $n = 1$, is satisfied by

$$v = r^{-\frac{3}{2}} \sin(qr + a) - qr^{-\frac{1}{2}} \cos(qr + a).$$

Examples.

(1) Prove the reduction of the differential equation

$$(i.) (1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = 0 \text{ to } \frac{d^2 y}{dt^2} = 0, \text{ when } x = \sin t;$$

$$(ii.) (a^2 + x^2) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = 0 \text{ to } \frac{d^2 y}{dt^2} = 0, \text{ when } x = a \tan t;$$

$$(iii.) x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + n^2 y = 0 \text{ to } \frac{d^2 y}{dt^2} + n^2 y = 0, \text{ when } x = e^t.$$

$$(iv.) x^{2n} \frac{d^n y}{dx^n} \pm a^n y = 0 \text{ to } b^n \frac{d^n u}{dz^n} \pm u = 0,$$

when $x/a = e^t = b/z$ and $y = x^{n-1}u$.

(2) Given $y = \sin(m\theta + a)$, and $x = \sin \theta$, prove that

$$(1 - x^2)y'' - xy' + m^2 y = 0;$$

and that, when $x = 0$,

$$y^{(2n-1)} = m(1^2 - m^2)(3^2 - m^2) \dots \{(2n-3)^2 - m^2\} \cos a,$$

$$y^{(2n)} = -m^2(2^2 - m^2)(4^2 - m^2) \dots \{(2n-2)^2 - m^2\} \sin a.$$

*(3) Denoting the Schwartzian derivative by (s, x) ,

$$(i.) (s, x) = -\left(\frac{ds}{dx}\right)^2 (x, s) = p' - \frac{1}{2}p^2, \text{ if } p = \frac{d \log s'}{dx};$$

$$(ii.) (s, x) = \left(\frac{dy}{dx}\right)^2 \{(s, y) - (x, y)\};$$

$$(iii.) \left(\frac{es+f}{Es+F} \frac{ax+b}{Ax+B}\right) = \left(s, \frac{ax+b}{Ax+B}\right) = \frac{(Ax+B)^4}{(aB-Ab)} (s, x).$$

- *(4) Prove that the linear differential equation of the third order $R_0 y''' + 3R_1 y'' + 3R_2 y' + R_3 y = 0$, where the R 's are functions of x , is reduced to the form $u''' + 3P_2 u' + P_3 u = 0$, by the substitution $u = y \exp \int R_1 dx / R_0$; and this again to the *canonical* form

$$\frac{1}{v} \frac{d^3 v}{dz^3} + \Theta = 0,$$

by the substitution $v = uz'$, where z' is given by the differential equation

$$(z, x) = \frac{3}{2} P_2, \text{ and } z'^3 \Theta = P_3 - \frac{3}{2} P_2'.$$

- *(5) Verify that $y = AFx + BF(-x)$ satisfies

$$(i.) \frac{1}{y} \frac{d^2 y}{dx^2} = 2 \operatorname{cosec}^2 x + \cot^2 a, \text{ or } 2 \operatorname{cosech}^2 x + \coth^2 a,$$

if $Fx = (\cot x + \cot a)e^{-x \cot a}$, or $(\coth x + \coth a)e^{-x \coth a}$;

$$(ii.) \frac{1}{y} \frac{d^2 y}{dx^2} = 6 \operatorname{cosec}^2 x + \cot^2 a, \text{ or } 6 \operatorname{cosech}^2 x + \coth^2 a,$$

$$\text{if } Fx = \frac{d}{dx}(\cot x + \cot b)e^{-x \cot a}, \dots$$

$$\text{where } \cot b = \frac{1}{3} \cot a - \frac{2}{3} \tan a.$$

- *(6) Prove that, if $x = (az + b)/(Az + B)$,

$$\frac{d^n y}{dx^n} = \frac{(Az + B)^{n+1}}{(Ab - aB)^n} \frac{d^n}{dz^n} \{ (Az + B)^{n-1} y \},$$

$$\text{or } \frac{(Az + B)^n}{(Ab - aB)^n} \frac{d^{n-1}}{dz^{n-1}} \left\{ (Az + B)^n \frac{dy}{dz} \right\}.$$

- *(7) Show how to eliminate the constants from the relation $y = (ax^2 + 2bx + c)/(Ax^2 + 2Bx + C)$, and generally from the relation

$$y = (ax^n + bx^{n-1} + \dots)/(Ax^n + Bx^{n-1} + \dots)$$

(MacMahon, *Phil. Mag.* June, 1887.)

89. Geometrical Illustrations of Successive Differentiation. Curvature.

The *curvature* at any point of a plane curve is defined to be the rate at which the curve is bending, or the rate at which the tangent is revolving, estimated per unit length of the curve. (Leibnitz, 1686.)

The angle (in circular measure) between the tangents (or normals) at the ends of a finite arc of a plane curve is called the *whole curvature* of the arc, and the ratio of the whole curvature to the length of the arc is called the *average curvature* of the arc.

Suppose that, in going from a point P to an adjacent point p along an arc of length Δs , the tangent (or normal) has turned through the angle $\Delta\psi$; then $\Delta\psi$ is the whole curvature, and $\Delta\psi/\Delta s$ is the average curvature of the arc Pp ; and making p move up on the curve to coincidence with P , the curvature at P is $\lim \Delta\psi/\Delta s = d\psi/ds$.

To measure curvature a circle is employed, because the curvature of a circle is the same at every point, and by varying the radius the circle may be made to have any required curvature.

For if Pp is the arc of a circle, then $\Delta s/\Delta\psi$ is always equal to the radius (§ 16), and therefore the curvature of a circle, measured by $\Delta\psi/\Delta s$, is the reciprocal of the radius.

Railway engineers speak of a curve of 1° , 2° , ..., meaning a curve of which a length of 100 feet has this whole curvature; and thus the radius of a 1° curve is $100 \times 180 \div \pi = 5730$ feet, of a 2° curve is 2865 feet, of a 3° curve is 1910 feet, and so on.

The *circle of curvature* or the *osculating circle* at any point of a curve is defined to be the circle which touches the curve at the point and has the same curvature.

90. Another way of discussing curvature is to consider a curve $P_1P_2P_3 \dots$ as the limit of a polygon formed by a number of equal short chords (or tangents), like the links of a chain or the carriages of a railway train on a curve (fig. 35).

If perpendiculars to the chords are drawn through their middle points, these lines will form a new polygon $Q_1Q_2Q_3 \dots$ such that any point Q_2 is the centre of a circle either passing through three consecutive points P_1, P_2, P_3 , or else touching two consecutive chords (or tangents) P_1P_2, P_2P_3 ; and the end of thread, wrapped on the polygon $Q_1Q_2Q_3 \dots$, can be made to pass through the points P_1, P_2, P_3, \dots , or else touch the chords P_1P_2, P_2P_3, \dots , on being wound on or unwound.

In the limit when the points P_1, P_2, P_3 are taken close together on the curve, the polygon $Q_1Q_2Q_3 \dots$ becomes a continuous curve, the locus of the centres of curvature, called the *evolute* of the curve $P_1P_2P_3 \dots$, and touched by all the normals of the curve P .

91. Any small arc Pp may be considered as ultimately coincident with the arc of the circle of curvature, and therefore Q , the point of intersection of the normal at p with the normal at P (fig. 35), is ultimately the *centre of curvature* at P ; and then QP is called the *radius of curvature*; and denoting it by ρ , then $\rho = \lim \Delta s / \Delta \psi = ds / d\psi$.

The curvature at any point is therefore $1/\rho$ or $d\psi/ds$.

$$\text{Now (§ 5)} \quad \tan \psi = \frac{dy}{dx}, \text{ or } \psi = \tan^{-1} \frac{dy}{dx};$$

$$\text{therefore (§ 25)} \quad \frac{d\psi}{dx} = \frac{d^2y}{dx^2} / \left(1 + \frac{dy^2}{dx^2}\right),$$

$$\text{also (§ 9)} \quad \frac{ds}{dx} = \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)};$$

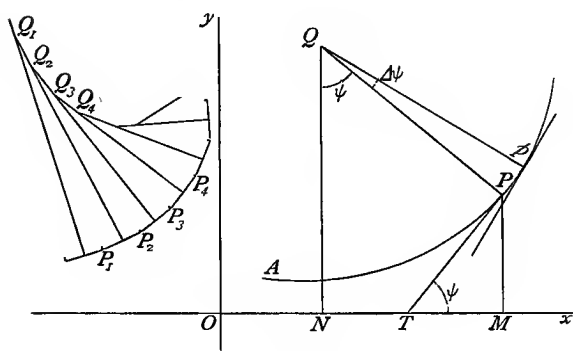


Fig. 35

and therefore, with x for independent variable,

$$\rho = \frac{ds}{d\psi} = \frac{ds}{dx} \bigg/ \frac{d\psi}{dx} = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}} \bigg/ \frac{d^2y}{dx^2}$$

the expression for ρ in terms of y , $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

92. Taking t as the independent variable, then

$$\psi = \tan^{-1} \frac{dy}{dx} = \tan^{-1} \left(\frac{dy/dt}{dx/dt} \right) \quad (\S 11),$$

$$\text{and } (\S 12) \quad \frac{d\psi}{dt} = \frac{\frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{d^2x}{dt^2} \frac{dy}{dt}}{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}};$$

$$\text{also } (\S 10) \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)},$$

$$\text{therefore} \quad \rho = \left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)^{\frac{3}{2}} \bigg/ \left(\frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{d^2x}{dt^2} \frac{dy}{dt}\right).$$

We have now *changed the independent variable* from x to t in the expression for ρ .

Since
$$v^2 = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2},$$

and
$$\frac{dx}{dt} = v \cos \psi, \quad \frac{dy}{dt} = v \sin \psi,$$

therefore
$$\rho = v^2 \left(\cos \psi \frac{d^2 y}{dt^2} - \sin \psi \frac{d^2 x}{dt^2} \right),$$

or
$$\frac{v^2}{\rho} = \cos \psi \frac{d^2 y}{dt^2} - \sin \psi \frac{d^2 x}{dt^2},$$

the *normal component* of the acceleration of P (fig. 34 ii.).

The *tangential component* of the acceleration

$$\begin{aligned} \cos \psi \frac{d^2 x}{dt^2} + \sin \psi \frac{d^2 y}{dt^2} \\ = \frac{1}{v} \left(\frac{dx}{dt} \frac{d^2 x}{dt^2} + \frac{dy}{dt} \frac{d^2 y}{dt^2} \right) = \frac{dv}{dt} = v \frac{dv}{ds} = \frac{d\frac{1}{2}v^2}{ds}, \end{aligned}$$

the same as for rectilinear motion; and v^2/ρ , the normal acceleration, is the same as for motion in the circle of curvature (§ 82).

For instance, in discussing the motion of a projectile in a resisting medium, in which the resistance acts in the direction opposite to motion, we begin by resolving normally, so as to eliminate the resistance; and then

$$v^2/\rho = g \cos \psi.$$

Now $v = ds/dt$, and $\rho = -ds/d\psi$, the negative sign appearing because ψ is always diminishing as s increases; so that $v^2 d\psi/ds = v dv/dt = -g \cos \psi$.

Put $\tan \psi = p$, and $v \cos \psi = u = dx/dt$, the horizontal component of the velocity; then

$$\frac{dt}{dp} = -\frac{u}{g}, \text{ and } \frac{dx}{dp} = -\frac{u^2}{g},$$

fundamental equations in the theory of the motion of a projectile in a resisting medium.

93. Changing to polar coordinates by putting

$$x = r \cos \theta, y = r \sin \theta, \text{ then by } \S 81,$$

$$\frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{d^2x}{dt^2} \frac{dy}{dt} = \frac{dr}{dt} \left(2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right) - r \left(\frac{d^2r}{dt^2} - r \frac{d\theta^2}{dt^2} \right),$$

and
$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = \frac{dr^2}{dt^2} + r^2 \frac{d\theta^2}{dt^2};$$

whence we find the value of the radius of curvature in terms of the polar coordinates, and their derivatives with respect to t .

Making θ the independent variable by putting $t = \theta$, then

$$d\theta/dt = 1, d^2\theta/dt^2 = 0;$$

and
$$\rho = \left(\frac{dr^2}{d\theta^2} + r^2 \right)^{\frac{3}{2}} / \left(r^2 + 2 \frac{dr^2}{d\theta^2} - r \frac{d^2r}{d\theta^2} \right);$$

so that, at the origin O , where $r = 0$, $\rho = \frac{1}{2} dr/d\theta$.

On replacing r by $1/u$,

$$\rho = \left(\frac{du^2}{d\theta^2} + u^2 \right)^{\frac{3}{2}} / \left(\frac{d^2u}{d\theta^2} + u \right) u^3;$$

and since
$$\frac{1}{p^2} = \frac{du^2}{d\theta^2} + u^2 \quad (\S 23),$$

therefore the chord of the circle of curvature through O is

$$\begin{aligned} 2\rho \sin \phi &= 2\rho p/r, \\ &= 2 \left(\frac{du^2}{d\theta^2} + u^2 \right) / \left(\frac{d^2u}{d\theta^2} + u \right) u^2; \end{aligned}$$

and therefore
$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\rho \sin^3 \phi}.$$

Since
$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{p^3} \frac{dp}{du},$$

therefore the chord of curvature through O is

$$-2p du / (u^2 dp) = 2p dr / dp;$$

and the diameter of curvature is $2r dr / dp$.

94. Taking s , the arc of the curve, for the independent variable, then since (§ 9)

$$\frac{dx}{ds} = \cos \psi, \quad \frac{dy}{ds} = \sin \psi;$$

therefore differentiating with respect to s ,

$$\frac{d^2x}{ds^2} = -\sin \psi \frac{d\psi}{ds} = -\frac{dy}{ds} \frac{1}{\rho},$$

$$\frac{d^2y}{ds^2} = \cos \psi \frac{d\psi}{ds} = \frac{dx}{ds} \frac{1}{\rho}.$$

Squaring and adding,

$$\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 = \frac{1}{\rho^2};$$

and

$$\rho = -\frac{dy}{ds} / \frac{d^2x}{ds^2} = \frac{dx}{ds} / \frac{d^2y}{ds^2};$$

also

$$\frac{1}{\rho} = \frac{d \sin \psi}{dx} = -\frac{d \cos \psi}{dy};$$

and

$$\cos \psi = \rho \frac{d^2y}{ds^2}, \quad \sin \psi = -\rho \frac{d^2x}{ds^2}.$$

If α, β denote the coordinates of Q , the *centre of curvature* at P ,

$$\alpha = x - \rho \sin \psi = x + \rho^2 \frac{d^2x}{ds^2}, \quad \beta = y + \rho \cos \psi = y + \rho^2 \frac{d^2y}{ds^2}.$$

With x for independent variable, the expressions are not so symmetrical; for

$$x - \alpha = \rho \sin \psi = \frac{ds}{d\psi} \frac{dy}{ds} = \frac{dy}{dx} / \frac{d\psi}{dx} = \frac{dy}{dx} \left(1 + \frac{dy^2}{dx^2}\right) / \frac{d^2y}{dx^2},$$

$$y - \beta = -\rho \cos \psi = -\frac{ds}{d\psi} \frac{dx}{ds} = -1 / \frac{d\psi}{dx} = -\left(1 + \frac{dy^2}{dx^2}\right) / \frac{d^2y}{dx^2}.$$

We may write
$$\beta = \frac{1}{2} \frac{d^2(x^2 + y^2)}{dx^2} / \frac{d^2y}{dx^2},$$

and by symmetry,
$$\alpha = \frac{1}{2} \frac{d^2(x^2 + y^2)}{dy^2} / \frac{d^2x}{dy^2}.$$

95. *The Evolute and Involute.*

Since $\alpha = x - \rho \sin \psi$, $\beta = y + \rho \cos \psi$;

then differentiating with respect to s ,

$$\frac{d\alpha}{ds} = \frac{dx}{ds} - \frac{d\rho}{ds} \sin \psi - \rho \cos \psi \frac{d\psi}{ds} = -\frac{d\rho}{ds} \sin \psi,$$

$$\frac{d\beta}{ds} = \frac{dy}{ds} + \frac{d\rho}{ds} \cos \psi - \rho \sin \psi \frac{d\psi}{ds} = \frac{d\rho}{ds} \cos \psi,$$

and therefore

$$\frac{d\beta}{d\alpha} = -\cot \psi = \tan(\tfrac{1}{2}\pi + \psi).$$

The locus of (α, β) the centre of curvature Q , is called the *evolute* of the curve AP (§ 90); and the preceding equation shows that the tangent to the evolute at Q is QP , the normal to the curve at P (fig. 35).

Also
$$\frac{d\alpha^2}{ds^2} + \frac{d\beta^2}{ds^2} = \frac{d\rho^2}{ds^2}, \text{ or } \frac{d\sigma^2}{ds^2} = \frac{d\rho^2}{ds^2},$$

if σ denotes the length of the arc of the evolute, measured from a fixed point.

Therefore
$$\frac{d\sigma}{ds} = \pm \frac{d\rho}{ds},$$

and

$$\sigma = a \text{ constant} \pm \rho.$$

(i.) Suppose $\sigma = \rho - l$; then the curve AP can be described by the end of a thread unwrapped from the corresponding arc BQ of the evolute (fig. 36, i.).

(ii.) Suppose $\sigma = l - \rho$; then the curve AP can be described by the end of a thread which is wrapped on the corresponding arc BQ of the evolute (fig. 36, ii.).

In each case, σ denotes the length of the arc BQ of the evolute, and $l = AB$, the radius of curvature at A , and $\rho = PQ$, the radius of curvature at P .

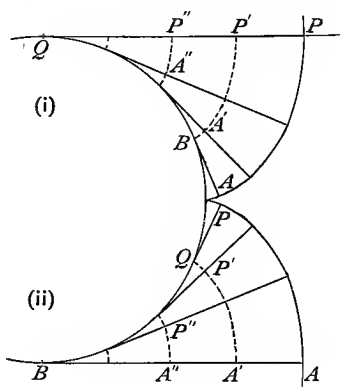


Fig. 36

Relatively to the curve BQ , the curve AP is called an *involute*; and by varying the length of the thread, any number of involutes $A'P'$, $A''P''$, can be obtained, all *parallel* curves, like the rails of a railway; but while a curve AP has only one evolute BQ , a curve BQ has any number of involutes; and a system of parallel curves have the same evolute.

Thus the tractrix (§ 36) is an involute of the catenary.

96. *Circle of Curvature*.—Another way of obtaining the circle of curvature at a point P of a curve is to regard it as the ultimate position of the circle which touches the curve at P and cuts the curve again at a consecutive point P' , when P' is moved up to coincidence with P .

For example, suppose the curve is a conic section, an ellipse (fig. 37), and let the circle cut the ellipse again at Q .

Produce QP' to meet the tangent at P in T ; then from a property of the circle, $TP^2 = TP' \cdot TQ$; and from a pro-

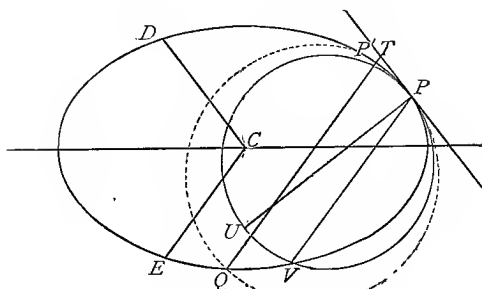


Fig. 37

perty of the ellipse, $\frac{TP^2}{TP' \cdot TQ} = \frac{CD^2}{CE^2}$, where CD, CE are the semidiameters of the ellipse parallel to TP, TQ .

Therefore $CD = CE$, and CD, CE therefore make equal angles with the axes of the ellipse; as also TP, TQ .

Now let P' approach to coincidence with P ; the circle becomes the circle of curvature at P , and TQ becomes PV , the common chord of the ellipse and its circle of curvature, and PV is therefore an *isoclinal* chord to PT , or equally inclined with the tangent PT to the axes.

Hence to construct the circle of curvature of an ellipse at P , draw the tangent PT and then the isoclinal chord PV ; PV will be the common chord of the ellipse and its circle of curvature; draw VU at right angles to PV to meet the normal at P in U , then PU will be the diameter of curvature at P .

The same construction holds for the parabola and hyperbola.

Also $PV = \text{lt } TQ = \text{lt } TP^2 / TP'$,

a relation which gives the chord of curvature PV in any direction and for any curve.

By changing the origin to any point P on a curve, and by taking the tangent PT and normal PU as coordinate axes of x and y , the equation of the curve may be written

$$y = \frac{1}{2}(ax^2 + 2hxy + by^2) + \text{higher powers};$$

and now $\text{lt } 2y/x^2 = \text{lt}(a + 2hy/x + by^2/x^2 + \dots) = a$, since $\text{lt } y/x = 0$; so that $a = 1/\rho$, where ρ is the radius of curvature at P .

97. Evolute of the Parabola.

Let x, y be the coordinates of a point P on the parabola $y^2 = 2lx$; and let α, β be the coordinates of Q , the centre of curvature at P ; to determine the locus of Q , the evolute of the parabola (fig. 38).

Differentiating the equation of the parabola with respect to x ,

$$2y \frac{dy}{dx} = 2l, \text{ or } \frac{dy}{dx} = \frac{l}{y};$$

and differentiating again

$$\frac{d^2y}{dx^2} = -\frac{l}{y^2} \frac{dy}{dx} = -\frac{l^2}{y^3}.$$

Therefore (§ 94)

$$x - \alpha = -\frac{l}{y} \left(1 + \frac{l^2}{y^2}\right) \bigg/ \frac{l^2}{y^3} = -\frac{y^2}{l} - l = -2x - l, \text{ or } \alpha = l + 3x;$$

$$\text{and } y - \beta = \left(1 + \frac{l^2}{y^2}\right) \bigg/ \frac{l^2}{y^3} = \frac{y^3}{l^2} + y, \text{ or } \beta = -\frac{y^3}{l^2}.$$

$$\text{Conversely } x = \frac{1}{3}(\alpha - l), \quad y = -(l^2\beta)^{\frac{1}{3}};$$

$$\text{and since } y^2 = 2lx,$$

$$\text{therefore } l^{\frac{2}{3}}\beta^{\frac{2}{3}} = \frac{2}{3}l(\alpha - l),$$

$$l\beta^2 = \frac{8}{27}(\alpha - l)^3,$$

the equation of the evolute, which is therefore a semi-cubical parabola (§ 58).

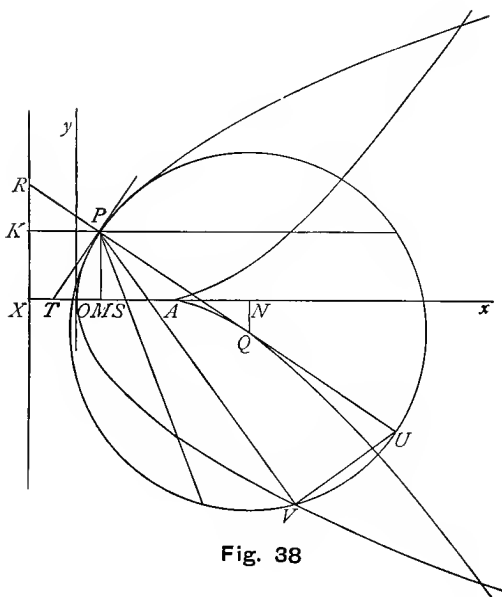


Fig. 38

Also
$$\rho = -\left(1 + \frac{l^2}{y^2}\right)^{\frac{3}{2}} / \frac{l^2}{y^3} = -\frac{(l^2 + y^2)^{\frac{3}{2}}}{l^2},$$

the negative sign appearing because d^2y/dx^2 is negative; therefore, changing the sign,

$$PQ = (l^2 + y^2)^{\frac{3}{2}} / l^2 = PG^3 / l^2 = l \operatorname{cosec}^3 \psi,$$

the normal at P meeting the axis of the parabola in G .

The arc of the evolute

$$AQ = PQ - OA = (l^2 + y^2)^{\frac{3}{2}} / l^2 - l,$$

which can also be expressed in terms of x , or α , or β .

This explains why the semi-cubical parabola can be rectified (§ 58), and in fact this was the first curve of which the rectification was effected algebraically (by Neil, in 1657, *Phil. Trans.*, 1673.)

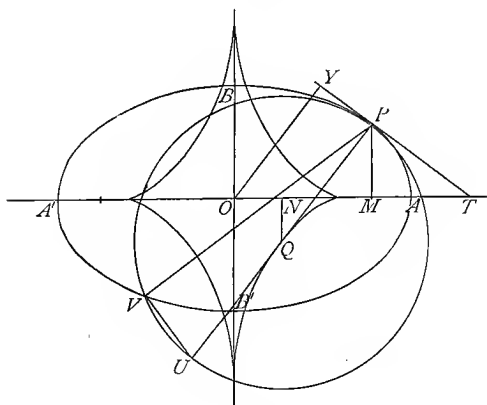


Fig. 39

98. *Evolute of the Ellipse and Hyperbola.*

Take θ , the excentric angle, as the independent variable in the ellipse (§§ 20, 51), then

$$x = a \cos \theta, \quad y = b \sin \theta;$$

$$\frac{dx}{d\theta} = -a \sin \theta, \quad \frac{dy}{d\theta} = b \cos \theta;$$

$$\frac{d^2x}{d\theta^2} = -a \cos \theta, \quad \frac{d^2y}{d\theta^2} = -b \sin \theta.$$

Thence
$$\rho = (a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{\frac{3}{2}} / ab$$

$$= a^2 b^2 \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} \right)^{\frac{3}{2}} = \frac{a^2 b^2}{p^3} = TP \cdot PV / OY.$$

Denoting the angle AOY by ω , then

$$\tan \omega = -\frac{dx}{dy} = \frac{a}{b} \tan \theta;$$

$$\sin \omega = \frac{a \sin \theta}{\sqrt{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)}}, \quad \cos \omega = \frac{b \cos \theta}{\sqrt{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)}}.$$

Therefore

$$\begin{aligned}x - a &= \rho \cos \omega = \frac{a^2 \sin^2 \theta + b^2 \cos^2 \theta}{a} \cos \theta \\&= a \cos \theta - \frac{a^2 - b^2}{a} \cos^3 \theta,\end{aligned}$$

or

$$\begin{aligned}aa &= (a^2 - b^2) \cos^3 \theta = (a^2 - b^2) x^3 / a^3 ; \\y - \beta &= \rho \sin \omega = \frac{a^2 \sin^2 \theta + b^2 \cos^2 \theta}{b} \sin \theta \\&= b \sin \theta + \frac{a^2 - b^2}{b} \sin^3 \theta,\end{aligned}$$

and

$$b\beta = -(a^2 - b^2) \sin^3 \theta = -(a^2 - b^2) y^3 / b^3.$$

Therefore, eliminating θ ,

$$(aa)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}},$$

the equation of the evolute of the ellipse.

Similarly the equation of the evolute of the hyperbola

$$(x/a)^2 - (y/b)^2 = 1,$$

is

$$(aa)^{\frac{2}{3}} - (b\beta)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}};$$

obtained by putting $x = a \cosh u$, $y = b \sinh u$;

when $aa = (a^2 + b^2) \cosh^3 u$, $b\beta = -(a^2 + b^2) \sinh^3 u$.

Examples.

- (1) Prove that the radius of curvature PQ of a parabola at P is $2PR$, where R is the point where the normal at P meets the directrix of the parabola.

Prove also that the evolute cuts the parabola at points which are the centres of curvature at $x = l$ of the parabola, at an angle $\text{cosec}^{-1} 3$.

- (2) Prove that in the rectangular hyperbola $xy = c^2$,

$$\alpha = \frac{3}{2}x + \frac{1}{2}y^3/c^2, \beta = \frac{3}{2}y + \frac{1}{2}x^3/c^2;$$

and that the equation of the evolute is

$$(\alpha + \beta)^{\frac{2}{3}} - (\alpha - \beta)^{\frac{2}{3}} = (4c)^{\frac{2}{3}}.$$

- (3) Prove that in the hyperbola $xy + Ax + By = 0$, the radius of curvature $\rho = \frac{1}{2}TP \cdot PV/OY$.

99. *The Figure and Size of the Earth.*

The size and figure of the Earth is inferred from the measured length of a degree or minute of latitude at sea level; the length is found to vary slightly with the latitude, showing that the Earth is not exactly spherical; but the length of a mean sexagesimal minute of latitude is taken as one geographical, nautical, or sea mile (French *mille*), so that the length of a quadrant of a meridian is $90 \times 60 = 5400$ sea miles; and the length of the Admiralty measured mile being taken as 6080 feet, the circumference of the Earth is $360 \times 60 \times 6080$ feet.

On the Metric System the quadrant is divided centesimally, and the mean centesimal minute of latitude is taken as the kilometre; so that the quadrant is $10000 = 10^4$ kilometres, or 10^7 metres, or 10^9 centimetres. This would make the metre = 3·2832 feet; but as the metre is more nearly 3·2809 feet, this shows that the quadrant is about 10007 kilometres; but for practical purposes it is sufficient to take the round numbers.

The *globe terrestre au millionième* at the Paris Exhibition of 1889 was a sphere 40 metres in circumference and therefore 12·732 metres in diameter, representing the Earth, 40,000 kilometres in circumference, and 12,732 kilometres in diameter, on a scale of one-millionth.

In navigation the speed of a vessel is always measured in *knots* (French *nœuds*, German *knoten*, Dutch *knoopen*, Spanish *nudos*, Italian *nodi*), one knot being a speed of one sea mile or mean sexagesimal minute of latitude per hour; when the knots on the log line are spaced so that the number of knots which pass over the taffrail in half a minute give the speed in *knots*, the knots must then be $6080 \div 120 = 50\cdot7$ feet apart.

100. *Dip and Distance of the Horizon. Variation in Length of a Degree and Minute of Latitude.*

When we ascend vertically from sea level at A to a point T , the horizon in consequence of the curvature of the Earth is bounded by the tangent lines TP (fig. 5) and the angle TPM is called the *dip of the horizon*, while TP is called the *distance of the horizon*, being the distance at which a light at T is visible from sea level, in the absence of atmospheric refraction.

Supposing the Earth is exactly spherical, then the dip $TPM = AOP = \theta = \sec^{-1}(1 + h/R)$, if R denotes the radius of the Earth, and h the height of T above sea level; while the distance of the horizon

$$TP = R \tan \theta = \sqrt{(2hR + h^2)}.$$

The quantity h/R is so small in all places accessible to us that we may neglect $(h/R)^2$; and then $TP = \sqrt{(2hR)}$; while $\tan \theta = \sqrt{(2h/R)}$; so that we may put d the dip in sexagesimal seconds $= \frac{180 \times 60 \times 60}{\pi} \sqrt{\frac{2h}{R}}$.

With the metre as unit of length, $R = 10^7 \div \frac{1}{2}\pi$; so that, for a height of h metres,

$$d = \frac{180 \times 60 \times 60}{\pi} \sqrt{\frac{2h \times \frac{1}{2}\pi}{10^7}} = 648 \sqrt{(h/10\pi)} \\ = \log^{-1}(\tfrac{1}{2} \log h + 2.063);$$

while $TP = \sqrt{(2h \times 10^7 \div \frac{1}{2}\pi)} = 20000 \sqrt{(h/10\pi)}$ (metres),
 $= 20 \sqrt{(h/10\pi)} = \log^{-1}(\tfrac{1}{2} \log h + .5525)$ (kilometres).

With $R = 10^4 \div \frac{1}{2}\pi$ (kilometres), the surface of sky covered with a uniform canopy of cloud at a height of x kilometres will be (§ 60)

$$2\pi(R+x)x = 2\pi Rx(1+x/R) = 40,000 x,$$

(square kilometres) neglecting the fraction x/R , which is practically very small.

Defining the latitude of a place as the angular altitude of the celestial pole, measured from the horizontal plane of the sea level or surface of mercury at the place, a plane perpendicular to the plumb line (Sir W. Thomson, *Navigation*), then it is found by geodetic measurements that the length of a minute of arc of latitude on the Earth's surface in latitude ψ may be taken as $1 - c \cos 2\psi$, sea miles or kilometres, according as the minute is sexagesimal, or centesimal; so that in latitude ψ the radius of curvature of the meridian is

$$(1 - c \cos 2\psi) 5400 \div \frac{1}{2}\pi \text{ sea miles,}$$

$$\text{or} \quad (1 - c \cos 2\psi) 10000 \div \frac{1}{2}\pi \text{ kilometres.}$$

Denoting by R the mean radius of the Earth, and by a, b its equatorial and polar radii, then

$$ds/d\psi = R(1 - c \cos 2\psi),$$

$$dx/d\psi = -R(1 - c \cos 2\psi) \sin \psi,$$

$$dy/d\psi = R(1 - c \cos 2\psi) \cos \psi;$$

$$x/R = \int_{\psi}^{\frac{1}{2}\pi} (1 - c \cos 2\psi) \sin \psi d\psi = (1 + \frac{1}{2}c) \cos \psi - \frac{1}{6}c \cos 3\psi,$$

$$y/R = \int_0^{\psi} (1 - c \cos 2\psi) \cos \psi d\psi = (1 - \frac{1}{2}c) \sin \psi - \frac{1}{6}c \sin 3\psi;$$

$$\text{and} \quad a/R = 1 + \frac{1}{3}c, \quad b/R = 1 - \frac{1}{3}c;$$

so that $(a - b)/R = \frac{2}{3}c$, and $\frac{2}{3}c$ is called the *compression* of the Earth; and it is found by geodetic measurements that $c = 1/200$, about; so that the compression is $1/300$.

Thus with $R = 6366$ kilometres, we find

$$a - b = 6366 \div 300 = 21 \text{ kilometres, about;}$$

on the *Globe au millionième* this would amount to only 21 millimetres, and would be quite insensible.

The length of the corresponding minute of longitude is $x/R = (1 + \frac{1}{2}c) \cos \psi - \frac{1}{6}c \cos 3\psi$, miles or kilometres.

Examples on Curvature:

- (1) Denoting the length of the normal PG by n , and the radius of curvature by ρ , prove that in the
 - (i.) Rectangular hyperbola, $2\rho = \text{normal chord}$, as in the circle.
 - (ii.) Catenary, $\rho = n$, as in the circle.
 - (iii.) Tractrix, $\rho = a^2/n$, or $\rho n = a^2$.
 - (iv.) Cycloid, $\rho = 2n$, as in the parabola.
 - (v.) Conic, $\rho = n^3/l^2$.
 - (vi.) Sinusoid, $y/b = \sin x/a$, $\rho = (a^2 + b^2 - y^2)^{\frac{3}{2}}/ay$.
 - (vii.) In $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, $\rho = 3(axy)^{\frac{1}{3}} = 3p$ (Ex. 6, p. 36).
 - (viii.) In the curves (*Sumner lines*),
 $\sinh y/a = \tan a \sin x/a$, or $\cosh y/a = \sec \beta \cos x/a$;
 $\rho = a \operatorname{cosec} a \operatorname{cosec} x/a$, or $a \sin \beta \sec x/a$.
- (2) Prove that in $y/a = \log \sec x/a$, the chord of curvature parallel to the axis of y is $2a$, and $\rho/a = \cosh s/a$.
- (3) Prove that, if $x = a + 2bt + ct^2$, $y = A + 2Bt + Ct^2$, the point describes a parabola; and that ρ varies as $\sec^3 \psi$, where ψ is the angle the tangent makes with the line $Cx + Cy = 0$.
- (4) Determine ρ as a function of θ in the curve, an ellipse, described by the point
 $x = a + b \cos \theta + c \sin \theta$, $y = A + B \cos \theta + C \sin \theta$.
- (5) Prove that, with polar coordinates, in the curve
 - (i.) $r = a\theta$, $\rho = (a^2 + r^2)^{\frac{3}{2}}/(2a^2 + r^2)$.
 - (ii.) $r = a/\theta$, $\rho = r(r^2 + a^2)^{\frac{3}{2}}/a^3$.
 - (iii.) $r = a^\theta$, $\rho = Pg$.
 - (iv.) $r = a \sin \theta$, or $a \cos \theta$, $\rho = \frac{1}{2}a$.
 - (v.) $r = a \sec \theta$, or $\operatorname{cosec} \theta$, $\rho = \infty$.
 - (vi.) $r = a \operatorname{vers} \theta$, $\rho = \frac{2}{3}\sqrt{2(ar)^{\frac{1}{2}}}$.
 - (vii.) $r = a\sqrt{(\cos 2\theta)}$, or $a\sqrt{(\sin 2\theta)}$, $\rho = \frac{1}{3}a^2/r$.
 - (viii.) $r = a\sqrt{(\sec 2\theta)}$, or $a\sqrt{(\operatorname{cosec} 2\theta)}$, $\rho = r^3/a^2$.

(6) Prove that in the curve $r = a + b \cos \theta$,

$$\rho = (a^2 + 2ab \cos \theta + b^2)^{\frac{3}{2}} / (a^2 + 3ab \cos \theta + 2b^2);$$

and that in the conic $l/r = lu = 1 + e \cos \theta$,

$$\rho = l(1 + 2e \cos \theta + e^2)^{\frac{3}{2}} / (1 + e \cos \theta)^3 = l \operatorname{cosec}^3 \phi,$$

ϕ denoting the radial angle.

(7) Prove that the chord of curvature through O of

$$r^n = a^n \cos n\theta \text{ is } 2r/(n+1), \text{ and } \rho = Pg/(n+1).$$

*(8) Prove that the normal chord which divides the area of an ellipse most unequally is equally inclined to the axes.

*(9) Prove that in the curves $(x/a)^n \pm (y/b)^n = 1$, and $(x/a)^m = (y/b)^n$, ρ is respectively $2/(1-n)$ times and twice the radius of curvature of the tangent hyperbola $xy + Ax + By = 0$, at the point of contact.

*(10) Prove that the locus of the centre (a, β) of a rectangular hyperbola, touching the ellipse (§ 98), and having the same curvature, the axes of the ellipse and hyperbola being parallel, is given by

$$aa = (a^2 + b^2) \cos^3 \theta, \quad b\beta = (a^2 + b^2) \sin^3 \theta;$$

$$\text{or} \quad (aa)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}.$$

*(11) Prove that in the *lemniscate* $(x^2 + y^2)^2 = a^2(x^2 - y^2)$, or $r^2 = a^2 \cos 2\theta$, we may put

$$x = a \cos \theta \sqrt{(\cos 2\theta)}, \quad y = a \sin \theta \sqrt{(\cos 2\theta)};$$

and then

$$a = \frac{2}{3}a \cos^3 \theta \sqrt{(\sec 2\theta)}, \quad \beta = -\frac{2}{3}a \sin^3 \theta \sqrt{(\sec 2\theta)};$$

$$\text{and} \quad (\alpha^{\frac{2}{3}} + \beta^{\frac{2}{3}})^2 (\alpha^{\frac{2}{3}} - \beta^{\frac{2}{3}}) = \frac{4}{9}a^2.$$

*(12) The evolute of the equiangular spiral $r = c \exp(\theta \cot \alpha)$ is the same curve turned about O through an angle

$$\frac{1}{2}\pi + \tan \alpha \log \tan \alpha \text{ (radians).}$$

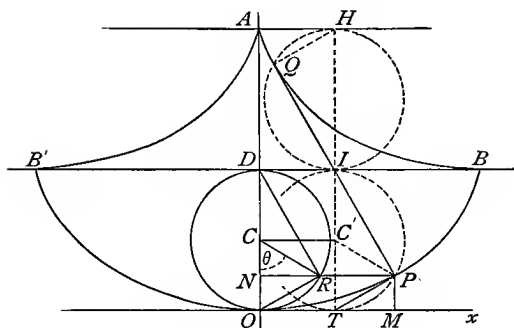


Fig.40

101. *The Cycloid and its Evolute. Cycloidal Oscillations.*

The cycloid, defined in § 21, is now drawn inverted, for purposes of dynamical illustration, being generated by the rolling of a circle on the under side of the horizontal line $B'B$ (fig. 40).

If the circle were to change to rolling on the upper side of Ox , then a cycloid as in fig. 9 would be described, cutting the other cycloid at right angles; thus the *orthogonal* trajectories of equal cycloids with base on Ox are equal cycloids with base on BB' .

With the new coordinate axes of fig. 40,

$$x = a(\theta + \sin \theta), \quad y = a \operatorname{vers} \theta;$$

so that
$$x = a \operatorname{vers}^{-1} y/a + \sqrt{(2ay - y^2)} \dots \dots \dots (1)$$

The curve described by P' when NP' is made equal to the arc OR is called *Roberval's companion of the cycloid*, but then $x = a\theta$, $y = a \operatorname{vers} \theta$; or $y = a \operatorname{vers} x/a$;
a curve of the same form as the *sinusoids* (§ 20)

$$y = a \cos x/a \text{ or } y = a \sin x/a.$$

IP is the normal and TP is the tangent at P , I being the centre of instantaneous rotation of the rolling circle; and now, as in § 21, $\psi = \frac{1}{2}\theta$, while $ds/d\theta = 2a \cos \frac{1}{2}\theta$;

so that $\dot{s} = 4a \sin \frac{1}{2}\theta = 4a \sin \psi \dots \dots \dots (2)$,
 s denoting the arc OP , which is therefore equal to $2TP$.

The value of ρ , the radius of curvature at P , can be obtained from the formula (§ 92)

$$\rho = \left(\frac{dx^2}{d\theta^2} + \frac{dy^2}{d\theta^2} \right)^{\frac{3}{2}} / \left(\frac{dx}{d\theta} \frac{d^2y}{d\theta^2} - \frac{d^2x}{d\theta^2} \frac{dy}{d\theta} \right),$$

and will be found to be $4a \cos \frac{1}{2}\theta$; but this can be obtained immediately from equation (2) by § 91; for

$$\rho = ds/d\psi = 4a \cos \psi = 4a \cos \frac{1}{2}\theta.$$

Therefore $PQ = 2PI$, if Q is the centre of curvature at P , and this shows that the evolute AQ is an equal cycloid.

For α, β denoting the coordinates of Q , then (§ 91)

$$\begin{aligned} \alpha &= x - \rho \sin \psi = a(\theta + \sin \theta) - 4a \cos \frac{1}{2}\theta \sin \frac{1}{2}\theta \\ &= a(\theta - \sin \theta), \end{aligned}$$

$$\begin{aligned} \beta &= y + \rho \cos \psi = a(1 - \cos \theta) + 4a \cos^2 \frac{1}{2}\theta \\ &= 4a - a \text{ vers } \theta; \end{aligned}$$

which proves that, as P describes the cycloid OPB , Q describes an equal cycloid AQB , produced by rolling a circle of radius a on the horizontal straight line through A .

This, the first problem of an evolute, was invented by Huyghens (1673), with the object of making a particle oscillate in a cycloid; for let the cycloid BOB' be cut out of some material and divided at DO , and $B'O$ be placed in the position AB , and OB in the position $B'A$, in a vertical plane with BB' horizontal; a particle at O hanging vertically from A by a fine thread of length OA will, if made to oscillate in the vertical plane of the figure, describe an arc of the cycloid BOB' .

102. *Isochronism of Cycloidal Oscillations.*

The peculiarity of these oscillations is that the *period* is the same for all arcs of oscillation; this property is called the *isochronism of the cycloid*.

For resolving tangentially at P ,

$$dv/dt = d\frac{1}{2}v^2/ds = -g \sin \psi = -gs/l,$$

if $l = 4a$, the length of the thread.

Integrating with respect to s , and denoting g/l by n^2 ,

$$\frac{1}{2}v^2 = \frac{1}{2}n^2(s_1^2 - s^2),$$

supposing P to be drawn aside from O through an arc s_1 , and then let go.

Therefore $v = ds/dt = -n\sqrt{(s_1^2 - s^2)}$,

the negative sign being taken because P begins moving towards O ; and

$$n \frac{dt}{ds} = -\frac{1}{\sqrt{(s_1^2 - s^2)}}, \quad n(t - \tau) = \int_s^{s_1} \frac{ds}{\sqrt{(s_1^2 - s^2)}} = \cos^{-1} \frac{s}{s_1},$$

or

$$s = s_1 \cos n(t - \tau),$$

supposing τ to denote the instant of time when P is let go.

If $\frac{1}{2}T$ denotes the time of oscillation from rest to rest, then $n(t - \tau)$ increases by π , while $t - \tau$ increases by $\frac{1}{2}T$, so that $nT = 2\pi$, and $s = s_1 \cos \{2\pi(t - \tau)/T\}$,

and

$$T = 2\pi\sqrt{(l/g)},$$

which is independent of s_1 , and therefore the same for all arcs of vibration; which proves the isochronism of the cycloid.

When s_1 , the *amplitude* of vibration, is small, the arc of vibration in the cycloid may be considered coincident with the arc of the circle of curvature at O , a circle of radius l and centre A ; so that the period of a small plane oscillation of a simple pendulum of length l is thus proved to be $2\pi\sqrt{(l/g)}$, the same as the period of revolution in a small horizontal circle (§ 82).

103. *Harmonic Vibrations.*

If a point P describes the circle (fig. 21) with constant velocity in the periodic time T , then

$$\theta/2\pi = (t - \tau)/T, \text{ or } \theta = n(t - \tau),$$

supposing τ is an instant of time when P is at A ;

and then $OM = x = a \cos \theta = a \cos \{2\pi(t - \tau)/T\}$.

The point M oscillates between A and A' in the time $\frac{1}{2}T$, and M is said to perform a *simple harmonic motion* (S.H.M.) ; and a is called the *amplitude*, T the *period*, and τ is called the *phase*, being one of the instants at which M is at A ; while n is variously called the *speed*, or *angular velocity*, (or *mean motion*, in Astronomy).

In a steam engine the piston M performs a simple harmonic motion very approximately while the crank P moves with constant velocity, the slight discrepancy being due to the obliquity of the connecting rod ; and in § 102, P makes a harmonic vibration on the cycloid.

If
$$x = a \cos \{2\pi(t - \tau)/T\},$$

then
$$d^2x/dt^2 = -4\pi^2x/T^2 = -n^2x,$$

so that the point M vibrates as if *attracted* to O with intensity proportional to the distance from O .

The small vibrations of elastic bodies producing musical notes are of this nature, whence these vibrations are called *harmonic*.

In the plane oscillations of a simple circular pendulum composed of a small plummet at the end of a fine thread of length l feet, the plummet oscillates in a circle ; and since $s = l\theta$, resolving tangentially,

$$\frac{d^2s}{dt^2} = -g \sin \theta = -g \sin \frac{s}{l}.$$

The complete integration of this equation of motion requires the *Elliptic Functions*; but when the oscillations are so small that we may replace $\sin s/l$ by its c.m. s/l , then

$$d^2s/dt^2 = -gs/l = -n^2s,$$

so that the pendulum has a S.H.M. of period $2\pi\sqrt{l/g}$.

Again, if in the motion of P on the ellipse (§ 51, figs. 8 and 22), we put

$$\theta = 2\pi(t - \tau)/T = n(t - \tau),$$

then $x = a \cos n(t - \tau)$, $y = b \sin n(t - \tau)$;

so that the motion of P is compounded of two S.H.M.'s in directions parallel to Ox and Oy ; and since

$$d^2x/dt^2 = -n^2x, \quad d^2y/dt^2 = -n^2y,$$

therefore P moves as if attracted to O with intensity proportional to the distance from O .

Here, in the most general case of projection,

$$x = a \cos n(t - \tau), \quad y = b \cos n(t - \tau'),$$

with different phases τ and τ' ; and thence

$$\sin nt = \left(\frac{x}{a} \cos n\tau' - \frac{y}{b} \cos n\tau \right) / \sin n(\tau - \tau'),$$

$$\cos nt = \left(-\frac{x}{a} \sin n\tau' + \frac{y}{b} \sin n\tau \right) / \sin n(\tau - \tau');$$

so that, squaring and adding,

$$\frac{x^2}{a^2} - 2\frac{xy}{ab} \cos n(\tau - \tau') + \frac{y^2}{b^2} = \sin^2 n(\tau - \tau'),$$

the equation of a system of ellipses; having the same *orthoptic* circle $x^2 + y^2 = a^2 + b^2$, and all inscribed in the rectangle bounded by $x = \pm a$, $y = \pm b$; the points of contact forming a parallelogram of constant perimeter $2\sqrt{(a^2 + b^2)}$, the sides of which are parallel to the diagonals of the rectangle.

We may for brevity write x and y for x/a and y/b , and put $\cos n(\tau - \tau') = c$; and now

$$\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}c,$$

the ambiguity of sign corresponding with the two directions in which P can move round the ellipse; and this is equivalent to

$$xy \mp \sqrt{(1-x^2)}\sqrt{(1-y^2)} = c,$$

$$\text{or} \quad x\sqrt{(1-y^2)} \pm y\sqrt{(1-x^2)} = \sqrt{(1-c^2)},$$

$$\text{or} \quad x^2 - 2cxy + y^2 = 1 - c^2.$$

In the corresponding case for hyperbolas we have

$$\cosh^{-1}x \pm \cosh^{-1}y = \cosh^{-1}c,$$

$$\text{or} \quad x^2 - 2cxy + y^2 = 1 - c^2,$$

giving hyperbolas circumscribing a rectangle; while

$$\sinh^{-1}x \pm \sinh^{-1}y = \cosh^{-1}c,$$

represents the conjugate hyperbolas; and

$$\cosh^{-1}x \pm \sinh^{-1}y, \text{ or } \sinh^{-1}x \pm \cosh^{-1}y = \sinh^{-1}c,$$

represent rectangular hyperbolas and their conjugates; all described under a *repulsion* from O proportional to the distance, when we put

$$x/a = \cosh \text{ or } \sinh n(t - \tau), \quad y/b = \cosh \text{ or } \sinh n(t - \tau'); \quad \text{so that}$$

$$d^2x/dt^2 = n^2x, \quad d^2y/dt^2 = n^2y.$$

With the component velocities of Ex. 13, p. 27, the general conic is described with component accelerations

$$d^2x/dt^2 = (h^2 - ab)x - bg + fh = (h^2 - ab)(x - \bar{x}),$$

$$d^2y/dt^2 = (h^2 - ab)y - af + gh = (h^2 - ab)(y - \bar{y}),$$

where \bar{x}, \bar{y} are the coordinates of the centre; and if

$$X = (h^2 - ab)x^2 - 2(bg - fh)x + f^2 - bc,$$

$$Y = (h^2 - ab)y^2 - 2(af - gh)y + g^2 - ac,$$

$$\text{then} \quad dt = dx/\sqrt{X} = -dy/\sqrt{Y};$$

so that the integral of this differential relation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

With two component S.H.M's of different periods,

$$x = a \cos n(t - \tau), \quad y = b \cos m(t - \tau');$$

and eliminating t ,

$$m \cos^{-1} x/a - n \cos^{-1} y/b = mn(\tau' - \tau) = \alpha,$$

a constant; curves exhibited practically, in Lissajous's method, by a spot of light on a screen, reflected from small mirrors on two tuning forks of corresponding periods, vibrating in perpendicular planes. (Ganot, *Physics*; George M. Hopkins, *Experimental Science*.)

The student may exercise himself in drawing simple curves of this nature: for instance, with $\alpha = 0, \frac{1}{2}\pi, \dots$ and with $m/n = 1, 2, 3, 4, \dots 2/3, 3/4, \dots$ (Clifford, *Kinematic*.)

Draw also the curves given by

$$dt = \sqrt{X} dx = \sqrt{Y} dy,$$

and

$$dt = dx = dy / \sqrt{Y}, \text{ or } \sqrt{Y} dy.$$

*104. *Intrinsic Equation of a Curve.*

Equation (2) (§ 101) connecting s and ψ is called the *intrinsic equation* of the cycloid.

Generally the relation $s = f\psi$ for any curve, connecting s the length of any arc measured from a fixed point, and ψ the whole curvature of the arc (§ 89), is called the *intrinsic equation* of the curve (Whewell, *Cambridge Phil. Trans.*, vol. 8); the equation is called *intrinsic* because it is independent of a system of coordinates.

Thus $s = a\psi$ is the intrinsic equation of a circle of radius a ; and $s = l \sin \psi$ is the intrinsic equation of a cycloid, generated by the rolling of a circle of diameter $\frac{1}{2}l$.

Supposing ψ continually to increase, the complete cycloid will be composed of a number of equal branches coming to a point on the straight line $BB\dots$ in what are called *cusps* (fig. 41 i.), where $\rho = ds/d\psi = l \cos \psi$ vanishes.

More generally the curve whose intrinsic equation is

$$s = l \sin m\psi$$

will consist of a number of equal branches with equidistant cusps B, B, \dots now arranged on a circle.

If $m < 1$, the curve lies outside the circle of cusps $BBB\dots$, and is found to be an *epicycloid*—that is, the curve described by a point on the circumference of a circle rolling outside a fixed circle (fig. 41 ii.).

If $m > 1$, the curve lies inside the circle of cusps $BBB\dots$, and is a *hypocycloid*—that is, the curve described by a point on the circumference of a circle rolling inside a fixed circle (fig. 41, iii.).

The points A, A, \dots on these curves midway between the cusps are called the *apses*; and in the cycloid the apses lie on a straight line, in an epicycloid the apses lie on a circle greater than the circle of cusps, in the hypocycloid on a lesser circle.

To describe a curve from its intrinsic equation, $s = f\psi$, we find its radius of curvature $\rho = ds/d\psi = f'\psi$; and starting from a point A when $\psi = 0$, we describe a series of successive short circular arcs of appropriate radius ρ and curvature $\Delta\psi$, and thus build up a close approximation to the shape of a curve.

In this way the centering of arches is described practically (called in French *anse de panier*), and curves of section of complicated surfaces; such as capillary surfaces, of which the equations cannot be integrated (Sir W. Thomson, *Capillary Attraction*).

In drawing the curves $s = l \sin m\psi$, where $\rho = ml \cos m\psi$, then as ψ increases from 0 to $\frac{1}{2}\pi/m$, ρ diminishes from ml to 0, and half a branch is described; and the remainder of the curve is formed of symmetrical repetitions.

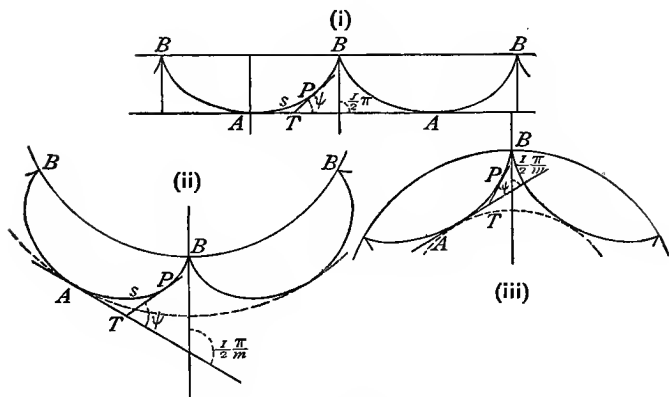


Fig.41

For practice the student may draw the epicycloids for $m = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots$; and the hypocycloids for $m = 2, 3, 4, \frac{3}{2}, \frac{4}{3}, \dots$; also their evolutes.

If $s = f\psi$ is the intrinsic equation of a curve, then

$$\sigma = ds/d\psi = f'\psi$$

is the intrinsic equation of the evolute (§ 92), measuring σ from the cusp where $ds/d\psi = 0$; and conversely

$$s = \int f\psi d\psi + C$$

is the intrinsic equation of an involute; and by varying C we obtain the parallel involutes.

For instance, if $s = l \sin m\psi$, then $\sigma = ds/d\psi = ml \cos m\psi$, which proves that the evolute of an epi- or hypo-cycloid is a similar epi- or hypo-cycloid, on a scale of m to 1.

Also $s = \frac{1}{2}a\psi^2$ is the intrinsic equation of the involute of a circle, the curve described by the hand in winding or unwinding a reel of thread (fig. 36); it is also the curve described by a smooth weight on a horizontal table in a railway carriage going at full speed, when entering a circular curve.

Examples.

(1) The intrinsic equation of the semicubical parabola is

(i.) $s = l(\sec^3 \psi - 1)$;

and of the parabola is (§ 97)

(ii.) $s = l \int \sec^3 \psi d\psi = \frac{1}{2} l \sec \psi \tan \psi + \frac{1}{2} l \log(\sec \psi + \tan \psi)$.

(iii.) $s = \frac{3}{4} a$ vers 2ψ , or $\frac{3}{4} a \sin 2\psi$ of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

(iv.) $s = a \tan \psi$ of the catenary.

(v.) $s = a \log \sec \psi$ of the tractrix.

(vi.) $s = a \log(\sec \psi + \tan \psi)$, or $\psi = \text{gd } s/a$,
or $\tan \frac{1}{2} \psi = \tanh \frac{1}{2} s/a$ of $y/a = \log \sec x/a$.

(vii.) $s = l \sin \frac{1}{3} \psi$ or l vers $\frac{1}{3} \psi$ of the cardioids

$$r = \frac{1}{4} l (1 \pm \cos \theta).$$

(viii.) $s = c \sec \alpha \{ \exp(\psi \cot \alpha) - 1 \}$ of $r = c \exp(\theta \cot \alpha)$.

(2) Draw the curves $\psi = m \sin s/a$, $m \sin^2 s/a^2$, $s/c + m \sin s/a$,

(i.) m small, (ii.) $m = 1$, (iii.) $m = \frac{1}{2} \pi$, (iv.) $m > \frac{1}{2} \pi$, ...

(3) Prove that the equation of the epicycloid $s = l \sin \frac{1}{2} \psi$
can be written $4(x^2 + y^2) - 3a^{\frac{2}{3}}x^{\frac{2}{3}} - a^2 = 0$, if $a = \frac{2}{3}l$.

105. *Envelopes.*

Since the normal of a curve is a tangent to the evolute (§§ 90, 95), the evolute is called the *envelope* of the normals, and the equation of the evolute is readily determined from this consideration; because the point of intersection of the normal at P with a consecutive normal is ultimately Q , the point of contact of QP on the evolute (fig. 35), and also the centre of curvature at P .

For instance, the equation of the normal to the ellipse at a point whose excentric angle is θ is (ex. 5, p. 35)

$$ax \sec \theta - by \operatorname{cosec} \theta - a^2 + b^2 = 0 \dots \dots \dots (i.)$$

Denoting this equation by $F\theta=0$, then, as explained below, to find the point of ultimate intersection of this normal with the consecutive normal, we must determine x and y from the equations

$$F\theta=0, \text{ and } F'\theta=0;$$

and the equation of the evolute is obtained by eliminating θ between these equations.

Here $F'\theta = ax \sec \theta \tan \theta + by \operatorname{cosec} \theta \cot \theta = 0 \dots\dots(ii.)$; and from (i.) and (ii.),

$$ax = (\alpha^2 - b^2) \cos^3 \theta, \quad by = -(\alpha^2 - b^2) \sin^3 \theta;$$

giving the coordinates of the centre of curvature; and

$$\text{eliminating } \theta, \quad (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (\alpha^2 - b^2)^{\frac{2}{3}},$$

the equation of the evolute (§ 98).

Similarly for the envelope of the normals of the hyperbola and of the parabola, in the forms

$$ax \operatorname{sech} u + by \operatorname{cosech} u = \alpha^2 + b^2,$$

and

$$y = m(x - l) - \frac{1}{2}m^3l,$$

for all values of u and m (§§ 97, 98).

Generally if $F\theta=0$ denotes the equation of any curve, involving x and y and a *parameter* θ , as it is called; keeping θ constant, a particular curve of the series is obtained; but by varying θ a series of curves is produced.

To find the points of ultimate intersection of the curve $F\theta=0$ with a consecutive curve of the series, suppose θ to receive a small increment $\Delta\theta$; then we must find x and y from the equations

$$F\theta=0, \text{ and } F(\theta + \Delta\theta)=0,$$

where $\Delta\theta=0$, ultimately; that is, from the equations

$$F\theta=0, \text{ and } \lim \{F(\theta + \Delta\theta) - F\theta\} / \Delta\theta = 0, \text{ or } F'\theta=0.$$

The locus of these points is called the *envelope* of the series of curves, and its equation is found by eliminating θ between the equations $F\theta=0$, and $F'\theta=0$.

This curve is called the *envelope* of the series, because each curve touches the envelope where it intersects the consecutive curve of the series.

For, take three consecutive curves of the series defined by the parameters $\theta + \Delta\theta$, θ , and $\theta - \Delta\theta$; and suppose the first and second to intersect in P , and the second and third in P' ; then P, P' are two consecutive points on the *envelope*, and also on the curve $F\theta = 0$, and therefore ultimately the envelope and a curve of the series have the same tangent where they meet.

Familiar instances of envelopes of straight lines, besides evolutes, are discussed in *Optics* with *caustic curves*, the envelopes of rays reflected or refracted at given curves or surfaces; as seen by reflexion on the surface of the water, or refracted through a glass of water.

Thus the caustic of rays reflected by a circle,

(i.) emanating from a point in the circumference of the circle, is a cardioid (ex. 11, p. 118);

(ii.) of parallel rays is the curve of ex. 3, p. 214;

(iii.) the caustic of rays emanating from a point, and refracted by a plane (§ 73), is the surface formed by the revolution of the evolute of an ellipse or hyperbola.

Examples.

(1) Find the envelope of a straight line of given length c , which moves with its ends on the coordinate axes.

(If the straight line makes an angle θ with the axis of x , it makes intercepts $c \cos \theta$, $c \sin \theta$ on the axes, and

$$x \sec \theta + y \operatorname{cosec} \theta = c.$$

Differentiating with respect to c ,

$$x \sec \theta \tan \theta - y \operatorname{cosec} \theta \cot \theta = 0;$$

and therefore $x = c \cos^3 \theta$, $y = c \sin^3 \theta$,

and eliminating θ (ex. 9 iv., § 14), $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$.)

- (2) Find the envelope of the parabolas of § 79, described by bodies projected from a fixed point O with given velocity V , but different elevation.

(Denoting by θ the variable elevation, and putting the *impetus* of projection $\frac{1}{2}V^2/g = h$, then the equation of the parabola is

$$y = x \tan \theta - \frac{1}{4}x^2 \sec^2 \theta / h;$$

and therefore the envelope is (fig. 33)

$$y = h - \frac{1}{4}x^2/h, \text{ or } x^2 = 4h(h - y);$$

the equation of the parabola HQ .

- (3) Prove that, referred to the centre of the wheel as origin, the envelope of the splashes of mud from a wheel, of radius a , on the side of an omnibus travelling with velocity $V = \sqrt{2gh}$, is

$$x^2 - a^2 = 4h(h - y).$$

- (4) Prove that the envelope of the ellipses $(x/a)^2 + (y/b)^2 = 1$

- (i.) when $a + b = c$, is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$;
- (ii.) when $a^2 + b^2 = c^2$, is $x \pm y = \pm c$;
- (iii.) when $ab = c^2$, is $xy = \pm \frac{1}{2}c^2$.

(Supposing a and b to be functions of some independent variable t , then, to find the envelope, differentiate with respect to t , treating a, b as variable, and x, y as constant;

then
$$\frac{x^2}{a^3} \frac{da}{dt} + \frac{y^2}{b^3} \frac{db}{dt} = 0;$$

and (i.) when $a + b = c$,
$$\frac{da}{dt} + \frac{db}{dt} = 0.$$

Multiplying the second equation by some *undetermined multiplier* λ , and adding it to the first equation; then

$$\left(\frac{x^2}{a^3} + \lambda\right) \frac{da}{dt} + \left(\frac{y^2}{b^3} + \lambda\right) \frac{db}{dt} = 0.$$

Now suppose λ so chosen that

$$x^2/a^3 + \lambda = 0; \text{ then } y^2/b^3 + \lambda = 0;$$

$$\text{and therefore } (x/a)^2 + (y/b)^2 + \lambda(a+b) = 0,$$

$$\text{or } 1 + \lambda c = 0, \lambda = -1/c.$$

$$\text{Therefore } a^3 = cx^2, b^3 = cy^2;$$

$$\text{and since } a + b = c,$$

$$\text{therefore } x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

Similarly for the other cases.)

(5) The envelope of the straight lines $x/a + y/b = 1$,

(i.) when $a + b = c$, is the parabola $\sqrt{x} + \sqrt{y} = \sqrt{c}$;

(ii.) when $a^2 + b^2 = c^2$, is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ (ex. 1);

(iii.) when $ab = c^2$, is the hyperbola $xy = \frac{1}{4}c^2$.

(6) The envelope of the curves

$$(x/a)^m + (y/b)^n = 1,$$

$$\text{where } a^n + b^n = c^n, \text{ is } x^{mn/(m+n)} + y^{mn/(m+n)} = c^{mn/(m+n)}.$$

(7) Prove geometrically that if a series of ellipses is described with equal major axes and one focus at a fixed point O , the envelope will be a parabola with focus at O if the other focus S of the ellipses moves on a fixed straight line, and the envelope will be an ellipse if S moves on a fixed circle.

(8) The envelope of the circles described on diameters which are the double ordinates of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ is the ellipse } \frac{x^2}{a^2 + b^2} + \frac{y^2}{b^2} = 1.$$

Prove also that the circles do not touch this envelope if the double ordinates are less than $2b^2/\sqrt{(a^2 + b^2)}$. This is the case with the circular sections of an ellipsoid, orthogonally projected on a parallel plane; as seen when looking at a model of an ellipsoid formed of parallel cardboard circles.

*General Examples on Successive Differentiation.

(1) Obtain an expression in the form of a determinant for the n^{th} differential coefficient of the quotient u/v .

(2) If $y = \tan^{-1} \frac{x \sin \alpha}{1 - x \cos \alpha}$, $\frac{d^n y}{dx^n} = \frac{(n-1)! \sin n(y+\alpha)}{(1-2x \cos \alpha + x^2)^{\frac{1}{2}n}}$.

(3) Prove that the differential equation

$$(i.) \frac{d^2 y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + p^2)y = 0$$

is satisfied by $y = e^{ax+b} \cos(px+q)$.

$$(ii.) \frac{d^4 y}{dx^4} - 2(a^2 - p^2) \frac{d^2 y}{dx^2} + (a^2 + p^2)^2 y = 0$$

by $y = A \cosh(ax+b) \cos(px+q)$

or $B \sinh(ax+b) \sin(px+q)$.

$$(iii.) \frac{d^2 x}{dt^2} + 2n \sinh a \frac{dy}{dt} + n^2 x = 0, \quad \frac{d^2 y}{dt^2} - 2n \sinh a \frac{dx}{dt} + n^2 y = 0,$$

by $x = a \cos ne^a(t-t_1) + b \cos ne^{-a}(t-t_2),$

$y = a \sin ne^a(t-t_1) - b \sin ne^{-a}(t-t_2);$

and then $\frac{d^4 x}{dt^4} + 2n^2 \cosh 2a \frac{d^2 x}{dt^2} + n^4 x = 0, \dots$

(4) Given the differential equations

$$\frac{dx}{dt} = n(x \cos 2\omega t + y \sin 2\omega t), \quad \frac{dy}{dt} = n(x \sin 2\omega t - y \cos 2\omega t),$$

$$\text{then } \frac{d^2 x}{dt^2} + 2\omega \frac{dy}{dt} - n^2 x = 0, \quad \frac{d^2 y}{dt^2} - 2\omega \frac{dx}{dt} - n^2 y = 0,$$

$$\text{and } \frac{d^4 x}{dt^4} + 2(2\omega^2 - n^2) \frac{d^2 x}{dt^2} + n^4 x = 0, \dots$$

Putting $\omega = n \cosh a$ or $n \cos a$, according as $\omega \geq n$, these equations are satisfied by

$$x = ae^{-\frac{1}{2}a} \cos n(te^a - \tau) + ae^{\frac{1}{2}a} \sin n(te^{-a} + \tau),$$

$$y = ae^{-\frac{1}{2}a} \sin n(te^a - \tau) - ae^{\frac{1}{2}a} \cos n(te^{-a} + \tau),$$

when $\omega > n$; and give the solution when $\omega < n$.

(5) Prove that $y = e^{ax} \sum_{s=0}^{s=m-1} x^s (A_s^{\cosh} \cosh px + B_s^{\sinh} \sinh px)$

satisfies the differential equation

$$\left\{ \left(\frac{d}{dx} - a \right)^2 \pm p^2 \right\}^m y = 0.$$

(6) (i.) $x = -\frac{a \cos mt}{m^2 - n^2} - \frac{bt \cos nt}{2n} + \frac{p \cosh mt}{m^2 + n^2} + \frac{q \sinh nt}{2n^2},$

or $x = -\frac{a \cos mt}{m^2 + n^2} - \frac{b \sin nt}{2n^2} + \frac{p \cosh mt}{m^2 - n^2} + \frac{qt \cosh nt}{2n}$

satisfy the differential equation

$$\frac{d^2 x}{dt^2} \pm n^2 x = a \cos mt + b \sin nt + p \cosh mt + q \sinh nt.$$

(ii.) $x = e^{at} \left(C_0 + C_1 t + \dots + C_{m-1} t^{m-1} + \frac{bt^m}{m!} \right) + \frac{ce^{pt}}{(p-a)^m}$

satisfies $\left(\frac{d}{dt} - a \right)^m x = be^{at} + ce^{pt};$

and write down the solution of

$$\left(\frac{d^2}{dt^2} \pm n^2 \right)^m x = a \cos mt + b \sin nt + p \cosh mt + q \sinh nt.$$

(7) Prove that $x^{\frac{2}{3}} \frac{d^3 y}{dx^3} = y$ is satisfied by

$$y = A e^{2\sqrt{x}} (1 - 2\sqrt{x}) +$$

$$B e^{-\sqrt{x}} \{ (1 + \sqrt{x}) \cos(\sqrt{3}\sqrt{x} + a) + \sqrt{3} \sin(\sqrt{3}\sqrt{x} + a) \}.$$

(8) Determine the most economical speed of a troopship, costing £400 a day for provisions and wages; given that the speed is 8 knots on a consumption of 50 tons of coal a day, costing 10s. a ton; and given that the consumption of coal per day varies as the cube of the speed.

(9) A cylindrical boiler is to be constructed of sheet iron of uniform thickness, with a longitudinal cylin-

drical flue of one-third of the external diameter. Prove that for given volume the weight of the boiler will be least when the length is two-thirds of the diameter.

Prove that with any given number of flues or tubes of given thickness, the diameters of which are proportional to the external diameter of the boiler, the weight for given volume will be a minimum when the weight of the cylindrical part and of the tubes is double the weight of the ends.

- (10) A number n of incandescent lamps each of internal resistance r ohms, are inserted in a single circuit of resistance R ohms. Show that in the most economical arrangement the number n should be the nearest integer to R/r , and that only about fifty per cent. of the energy can be utilized.
- (11) The strength of a rectangular beam of breadth b and depth d being proportional to bd^3 , and its stiffness to bd^3 , prove that the strongest and stiffest beams which can be cut from a circular log are such that the perpendicular from the corner of the cross section on the opposite diagonal will cut off a third or a fourth part of the diagonal.
- (12) Determine where it is economical to change from a cutting to a tunnel in constructing a level railway, crossing a ridge l yards broad and sloping on each side at an angle α ; taking the cost of tunnelling as £ A per linear yard, and of the cutting as £ B per cubic yard, the breadth of the railway being b yards, (i.) when the sides of the cutting are vertical, (ii.) when they slope at an angle β .

- (13) Determine geometrically where to stand in the road so as to throw a ball over a house with the least velocity.

Determine also where a ship comes within range of a fort on a cliff; and where the ship can return the fire; and where the ship can best batter the fort.

- (14) Prove that if w denotes the terminal velocity of a projectile and the resistance of the air is taken to vary as the velocity (§ 77), the trajectory for initial velocity V and elevation α is given in terms of the time of flight t by

$$x = \frac{Vw}{g} \cos \alpha (1 - e^{-gt/w}), \quad y = (\tan \alpha + \frac{w}{V} \sec \alpha)x - wt.$$

Prove also that, in a maximum range on an inclined plane through the point of projection, the angles of ascent and descent are complementary.

- (15) Show that the path of a rocket, if the velocity is maintained constant, is an inverted *catenary of equal strength* (ex. 2, p. 71); and that the range is proportional to the elevation.
- (16) Prove that, in epi- or hypo-cycloids,

$$\rho \frac{d^3 \rho}{ds^3} + 3 \frac{d\rho}{ds} \frac{d^2 \rho}{ds^2} = 0.$$

- (17) Prove that if c denotes the curvature $1/\rho$,

$$\left\{ 3 \frac{d^2 y}{dx^2} \frac{d^2 y}{dx^2} - 5 \left(\frac{d^3 y}{dx^3} \right)^2 \right\} / \left(1 + \frac{dy^2}{dx^2} \right)^4,$$

$$\text{or (§ 86)} \quad 36(4ac - 5b^2)/(1 + t^2)^4$$

$$= 3c \frac{d^2 c}{ds^2} - 5 \left(\frac{dc}{ds} \right)^2 + 9c^4 = \left\{ -3\rho \frac{d^2 \rho}{ds^2} + \left(\frac{d\rho}{ds} \right)^2 + 9 \right\} / \rho^4;$$

a quantity which vanishes for parabolas.

CHAPTER IV.

EXPANSION OF FUNCTIONS.

106. *Taylor's Theorem.*

In ordinary treatises on Algebra and Trigonometry the Binomial and Exponential Theorems are established, by means of which it is shown how to expand $(a+x)^m$, a^x , e^x , $\log(1+x)$, $\sin x$, $\cos x$, ... in ascending powers of x ; and we shall now show that all these and similar expansions are particular cases of one general Theorem called *Taylor's Theorem*, by means of which any function whatever can be expanded.

Taylor's Theorem is due to Dr. Brook Taylor, and was given by him in his "*Methodus Incrementorum Directa et Inversa*" in 1715.

The Theorem asserts that if $f(a+x)$ can be expanded in ascending positive integral powers of x , then

$$f(a+x) = fa + x f'a + \frac{x^2}{2!} f''a + \frac{x^3}{3!} f'''a + \dots + \frac{x^n}{n!} f^na + \dots,$$

$f'a$, $f''a$, $f'''a$, ... f^na , denoting the successive derivatives of fa with respect to a .

For assume that

$$f(a+x) = A_0 + xA_1 + x^2A_2 + x^3A_3 + \dots + x^nA_n + \dots$$

where the A 's are functions of a only, and not of x .

First put $x=0$, then $fa = A_0$.

Next differentiate successively with respect to x , and put $x=0$ after each differentiation; then

$$f'(a+x) = A_1 + 2xA_2 + \dots, f'a = A_1;$$

$$f''(a+x) = 2!A_2 + 2 \cdot 3xA_3 + \dots, f''a = 2!A_2;$$

$$f'''(a+x) = 3!A_3 + \dots, f'''a = 3!A_3;$$

$$\dots\dots\dots$$

$$f^n(a+x) = n!A_n + \dots, f^na = n!A_n.$$

$$\text{Therefore } A_0 = fa, A_1 = f'a, A_2 = \frac{1}{2!}f''a, A_3 = \frac{1}{3!}f'''a,$$

$$\text{and generally } A_n = \frac{1}{n!}f^na.$$

We have assumed here that

$$\frac{d}{da}f(a+x) = \frac{d}{dx}f(a+x);$$

and this is evident, if we consider that it is immaterial whether we change the function by increasing a or x .

Therefore, as Taylor's Theorem asserts,

$$f(a+x) = fa + x f'a + \frac{x^2}{2!}f''a + \frac{x^3}{3!}f'''a + \dots + \frac{x^n}{n!}f^na + \dots$$

As a simple illustration, consider the example of $fa = a^m$; then $f'a = ma^{m-1}$, $f''a = m(m-1)a^{m-2}$,; and therefore

$$\begin{aligned} f(a+x) &= (a+x)^m \\ &= a^m + ma^{m-1}x + \frac{m(m-1)}{2!}a^{m-2}x^2 + \dots, \end{aligned}$$

a verification of the *Binomial Theorem*.

Symbolical Form of Taylor's Theorem.

Employing the abbreviations of x_n for $\frac{x^n}{n!}$ and D for $\frac{d}{da}$, then Taylor's expansion can be written

$$f(a+x) = (1 + xD + x_2D^2 + \dots + x_nD^n + \dots)fa;$$

and treating the operator D as an algebraical quantity (§ 69), this is equivalent to

$$f(a+x) = e^{xD}fa, \text{ or } \exp(xD)fa,$$

which is called the *symbolical form* of Taylor's Theorem.

Maclaurin's Theorem.

Suppose $a=0$ in Taylor's Theorem ; then

$$fx = f0 + xf'0 + \frac{x^2}{2!}f''0 + \frac{x^3}{3!}f'''0 + \dots + \frac{x^n}{n!}f^n0 + \dots$$

which is called *Maclaurin's Theorem* ; but this theorem was first given by Stirling in 1717.

The meaning of f^n0 is that fx must be differentiated n times with respect to x , and then x put equal to 0 *after* the differentiation.

In Taylor's Theorem $f(a+x)$ is expanded in ascending powers of x , a part of the whole argument $a+x$ of the function $f(a+x)$; in Maclaurin's Theorem fx is expanded in powers of x , the whole argument of the function of x .

We might have proved Maclaurin's Theorem first, in the manner above in which Taylor's Theorem was obtained, and then have derived Taylor's Theorem by putting

$$fx = F(a+x) ;$$

and then, differentiating n times with respect to x ,

$$f^nx = F^n(a+x) ; \text{ and putting } x=0, f^n0 = F^na.$$

Substituting in Maclaurin's Theorem,

$$F(a+x) = Fa + xF'a + \frac{x^2}{2!}F''a + \dots + \frac{x^n}{n!}F^na + \dots,$$

which is Taylor's Theorem.

In fact the two theorems of Taylor and Maclaurin are the same when considered geometrically as the equation of a curve with different origins at a distance a apart on the axis of x .

107. *Application to the Expansion of Functions.*(1) Let $fx = \sin x$, then $f0 = 0$;also (§ 68) $f^n x = \sin(x + n\frac{1}{2}\pi)$, $f^n 0 = \sin \frac{1}{2}n\pi$;so that $f^{2n} 0 = \sin n\pi = 0$, $f^{2n+1} 0 = \sin(n\pi + \frac{1}{2}\pi) = (-1)^n$.

Therefore, by Maclaurin's Theorem,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

(2) Let $fx = \cos x$, then $f0 = 1$;and $f^n x = \cos(x + \frac{1}{2}n\pi)$, $f^n 0 = \cos \frac{1}{2}n\pi$;so that $f^{2n} 0 = (-1)^n$, $f^{2n+1} 0 = 0$.

Therefore, by Maclaurin,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{2n!} + \dots$$

(3) Let $fx = e^x$, then $f0 = 1$;also $f^n x = e^x$, and $f^n 0 = 1$.

Therefore

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Changing x into cx ,

$$e^{cx} = 1 + cx + \frac{c^2 x^2}{2!} + \dots + \frac{c^n x^n}{n!} + \dots$$

Again, suppose

$$fx = a^x, \quad \text{then } f0 = 1;$$

$$f^n x = a^x (\log a)^n, \text{ and } f^n 0 = (\log a)^n.$$

Therefore

$$a^x = 1 + x \log a + \frac{x^2 (\log a)^2}{2!} + \dots + \frac{x^n (\log a)^n}{n!} + \dots$$

This expansion is the same as the preceding, if $c = \log a$, since $a^x = e^{x \log a}$ (§ 31).(4) $\sinh x = \frac{1}{2}(e^x - e^{-x})$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$\begin{aligned}
 (5) \quad \cosh x &= \frac{1}{2}(e^x + e^{-x}) \\
 &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{2n!} + \dots
 \end{aligned}$$

These expansions might have been obtained independently by Maclaurin's Theorem.

(6) Let

$$f(1+x) = \log(1+x), \text{ then } f1 = 0;$$

$$f'(1+x) = \frac{1}{1+x}, \quad f'1 = 1;$$

$$f''(1+x) = -\frac{1}{(1+x)^2}, \quad f''1 = -1;$$

.....

$$f^n(1+x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \quad f^n1 = (-1)^{n-1}(n-1)!.$$

Therefore by Taylor's Theorem (here $a=1$)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n-1}x^n}{n} + \dots$$

We cannot expand $\log x$ in ascending powers of x , because if $fx = \log x$, then $f0, f'0, f''0, \dots$ are all infinite; the same applies to $\text{vers}^{-1}x$, $\exp(-1/x)$, $\exp(-1/x^2), \dots$

(7) From the preceding expansion,

$$\tanh^{-1}x = \frac{1}{2} \log \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2n+1}}{2n+1} + \dots$$

(8)

$$\sin(x+h) = \sin x \left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} \dots\right) + \cos x \left(h - \frac{h^3}{3!} + \frac{h^5}{5!} \dots\right)$$

$$\cos(x+h) = \cos x \left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} \dots\right) - \sin x \left(h - \frac{h^3}{3!} + \frac{h^5}{5!} \dots\right);$$

whence the expansions of $\sin h$ and $\cos h$ in ascending powers of h are inferred, from the formulas

$$\sin(x+h) = \sin x \cos h + \cos x \sin h, \quad \cos(x+h) = \dots$$

108. Some expansions can be derived from others by differentiation or integration; thus

$$\cos x = \frac{d \sin x}{dx}, \quad \cosh x = \frac{d \sinh x}{dx},$$

giving the expansion of $\cos x$ or $\cosh x$ when that of $\sin x$ or $\sinh x$ is known, and *vice versa*.

Again, by integration,

$$\begin{aligned} \log(1+x) &= \int_0^x \frac{dx}{1+x} = \int_0^x (1+x)^{-1} dx \\ &= \int_0^x (1 - x + x^2 - x^3 + \dots) dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \end{aligned}$$

$$\begin{aligned} \text{And } \tanh^{-1} \frac{x}{a} &= \int_0^x \frac{a dx}{a^2 - x^2} \\ &= \int_0^x \left(\frac{1}{a} + \frac{x^2}{a^3} + \frac{x^4}{a^5} + \dots \right) dx = \frac{x}{a} + \frac{x^3}{3a^3} + \frac{x^5}{5a^5} + \dots \end{aligned}$$

$$\begin{aligned} \text{Also } \tan^{-1} \frac{x}{a} &= \int_0^x \frac{a dx}{a^2 + x^2} \\ &= \int_0^x \left(\frac{1}{a} - \frac{x^2}{a^3} + \frac{x^4}{a^5} - \dots \right) dx = \frac{x}{a} - \frac{x^3}{3a^3} + \frac{x^5}{5a^5} - \dots, \end{aligned}$$

which is called *Gregorie's series*.

Putting $x/a = 1$, then

$$\tan^{-1} 1 = \frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots,$$

a slowly convergent series for the calculation of π .

When $x > a$, the series becomes divergent, and then

$$\tan^{-1} \frac{x}{a} = \frac{1}{2}\pi - \tan^{-1} \frac{a}{x} = \frac{1}{2}\pi - \frac{a}{x} + \frac{a^3}{3x^3} - \frac{a^5}{5x^5} + \dots$$

$$\begin{aligned} \text{Similarly, } \sin^{-1} \frac{x}{a} &= \int_0^x \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a} \int_0^x \left(1 - \frac{x^2}{a^2} \right)^{-\frac{1}{2}} dx \\ &= \int_0^x \left(1 + \frac{1}{2} \frac{x^2}{a^2} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^4}{a^4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^6}{a^6} + \dots \right) d \frac{x}{a} \end{aligned}$$

$$= \frac{x}{a} + \frac{1}{2} \frac{x^3}{3a^3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5a^5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^7}{7a^7} + \dots;$$

and $\sinh^{-1} \frac{x}{a} = \int_0^{\frac{x}{a}} \frac{dx}{\sqrt{(a^2 + x^2)}} = \frac{x}{a} - \frac{1}{2} \frac{x^3}{3a^3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5a^5} + \dots;$

or, without integration, we may assume that

$$\sin^{-1} x = A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + A_5 x^5 + \dots;$$

and then by differentiation

$$\begin{aligned} A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + 5A_5 x^4 + \dots \\ = (1 - x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2} x^2 + \frac{1 \cdot 3}{2 \cdot 4} x^4 + \dots, \end{aligned}$$

by the Binomial Theorem; and equating coefficients of like powers of x ,

$$A_1 = 1, A_2 = 0, A_3 = \frac{1}{2 \cdot 3}, A_4 = 0, A_5 = \frac{1 \cdot 3}{2 \cdot 4 \cdot 5}, \dots,$$

as before.

Put $x = 1$, then

$$\sin^{-1} 1 = \frac{1}{2} \pi = 1 + \frac{1}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} + \dots,$$

another series for the calculation of π ; or put $x = \frac{1}{2}$, then

$$\sin^{-1} \frac{1}{2} = \frac{1}{6} \pi = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 8} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 32} + \dots,$$

a more rapidly convergent series for π .

To expand any rational algebraical function of x , we resolve it, as for Integration, into its quotient and partial fractions, and then expand each partial fraction in powers of x , by the Binomial Theorem.

Any powers or products of sines or cosines of multiples of x , circular or hyperbolic, are expanded immediately, when we resolve them into sines or cosines of other multiples of x , as in Integration (§ 40).

109. Many expansions of functions are readily obtained, in Newton's manner, by forming the differential equation satisfied by the function, and then deducing by successive differentiation a recurring relation between the coefficients.

For instance, if $y = fx = \exp(a \sin^{-1}x)$,

then
$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0;$$

and differentiating n times, by Leibnitz's Theorem,

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} - (a^2+n^2)\frac{d^2y}{dx^2} = 0;$$

and now putting $x=0$,

$$f^{n+2}0 - (a^2+n^2)f^n0 = 0,$$

a recurring formula; whence, since $f0=1$, $f'0=a$, we deduce $f''0=a^2$, $f'''0=a(a^2+1^2)$, $f^{(4)}0=a^2(a^2+2^2)$,...

so that $fx = \exp(a \sin^{-1}x) = 1 + ax + a^2x_2 + a(a^2+1^2)x_3 + a^2(a^2+2^2)x_4 + a(a^2+1^2)(a^2+3^2)x_5 + \dots$

By expanding $\exp(a \sin^{-1}x)$ in powers of $a \sin^{-1}x$, and equating coefficients of a , a^2 , a^3 ,... we deduce the expansions in powers of x of $\sin^{-1}x$, $\frac{1}{2}(\sin^{-1}x)^2$, $\frac{1}{6}(\sin^{-1}x)^3$,...; thus

$$\begin{aligned}\sin^{-1}x &= x + x_3 + 3^2x_5 + 3^2 \cdot 5^2 \cdot x_7 + \dots, \\ \frac{1}{2}(\sin^{-1}x)^2 &= x_2 + 2^2 \cdot x_4 + 2^2 \cdot 4^2 \cdot x_6 + 2^2 \cdot 4^2 \cdot 6^2 \cdot x_8 + \dots, \\ \frac{1}{6}(\sin^{-1}x)^3 &= x_3 + (1^2+3^2)x_5 + (1^2+3^2+5^2)x_7 + \dots;\end{aligned}$$

and so on; the expansion of $\sin^{-1}x$ having been given already in § 108; and putting $x = \sin \theta$, we obtain the expansions of θ , $\frac{1}{2}\theta^2$, $\frac{1}{6}\theta^3$,... in powers of $\sin \theta$.

The versed sine, or rather the half versed sine, denoted by *havers*, is much used in Navigation; and

$$\text{havers } \theta = \frac{1}{2}(1 - \cos \theta) = \sin^2 \frac{1}{2}\theta;$$

so that the preceding expansions give θ , $\frac{1}{2}\theta^2$, $\frac{1}{6}\theta^3$,... in a series of powers of *havers* 2θ .

Then putting $2\theta = \phi$, we obtain

$$\frac{1}{8}\phi^2 = \frac{\text{havers } \phi}{2!} + \frac{2^2(\text{havers } \phi)^2}{4!} + \frac{2^2 \cdot 4^2(\text{havers } \phi)^3}{6!} + \dots;$$

$$\text{or } \frac{1}{2}\phi^2 = \text{vers } \phi + \frac{(\text{vers } \phi)^2}{6} + \frac{1 \cdot 2}{3 \cdot 5} \frac{(\text{vers } \phi)^3}{3} + \dots;$$

and differentiating both sides with respect to $\text{vers } \phi$,

$$\phi \cos \phi = 1 + \frac{1}{3} \text{vers } \phi + \frac{1 \cdot 2}{3 \cdot 5} (\text{vers } \phi)^2 + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} (\text{vers } \phi)^3 + \dots$$

In a similar manner the expansion of $\sin(m \sin^{-1} x + a)$ can be established, the differential equation and the recurring formula being (ex. 2 iii, p. 143; ex. 2, p. 185)

$$(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0$$

$$\text{and} \quad f^{n+2} 0 - (n^2 - m^2) f^n 0 = 0.$$

By differentiation we obtain the expansion of

$$(1-x^2)^{-\frac{1}{2}} \sin(m \sin^{-1} x + a);$$

and by putting $x = \sin \theta$, we obtain the expansion of $\sin(m\theta + a)$ in powers of $\sin \theta$.

Examples.

(1) Expand, in ascending (and descending) powers of x ,

$$\frac{1, x, x^2, \dots}{(x-a)(x-b)(x-c) \dots};$$

also in powers of $x-a$, or $x-b$, or $x-c$,

(Resolve into partial functions, of the form

$$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \dots;$$

then

$$\frac{A}{x-a} = -\frac{A}{a} \left(1 - \frac{x}{a}\right)^{-1} = -A \left(\frac{1}{a} + \frac{x}{a^2} + \frac{x^2}{a^3} + \dots + \frac{x^n}{a^{n+1}} + \dots\right),$$

$$\text{or} \quad = \frac{A}{x} \left(1 - \frac{a}{x}\right)^{-1} = A \left(\frac{1}{x} + \frac{a}{x^2} + \frac{a^2}{x^3} + \dots + \frac{a^{n-1}}{x^n} + \dots\right).$$

To expand in powers of $x-b$, put $x-b=z$; then $\frac{A}{x-a} = \frac{A}{z-a+b}$, which is expanded in powers of z , ascending or descending, as above.)

(2) Expand $fx/(a-x)^m$ in ascending powers of x , fx denoting a rational integral function of x .

Determine the remainder when fx is divided by $x-a$, $(x-a)(x-b)$, $(x-a)^2$, $(x-a)^3$,

(3) Prove that $e^{x \cos a} \cos(x \sin a)$

$$= 1 + x \cos a + \frac{x^2}{2!} \cos 2a + \dots + \frac{x^n}{n!} \cos na + \dots$$

(4) $e^{ax} \cos px$

$$= 1 + ax + \frac{a^2 x^2}{2!} \cos 2a (\sec a)^2 + \dots + \frac{a^n x^n}{n!} \cos na (\sec a)^n + \dots,$$

where $\tan a = p/a$; and determine the expansion of $e^{ax} \sin px$.

(5) $\cosh ax \cos px$

$$= 1 + \frac{a^2 x^2}{2!} \cos 2a (\sec a)^2 + \dots + \frac{a^{2n} x^{2n}}{2n!} \cos 2na (\sec a)^{2n} + \dots;$$

and write down the expansions of $\sinh ax \cos px$, $\cosh ax \sin px$, $\sinh ax \sin px$.

(6) Expand $\sin(mx+nh)$, $\cos(mx+nh)$, $f(mx+nh)$ by Taylor's Theorem in powers of h , n , m , or x .

*(7) Prove that (i.) $(\sin, \text{ or } \tan, \text{ or } \sin^{-1}, \text{ or } \tan^{-1}) \left(a \frac{d}{dx} \right)^{\sin} mx$
 $= (\sinh, \text{ or } \tanh, \text{ or } \sinh^{-1}, \text{ or } \tanh^{-1}) a m \frac{\cos}{-\sin} mx.$

$$(ii.) \tanh \frac{1}{2} h \frac{d}{dx} \{f(x+h) + fx\} = f(x+h) - fx.$$

$$(iii.) \exp \left(x^2 \frac{d}{dx} \right) fx = f \left(\frac{x}{1-x} \right).$$

*(8) From the relation $\log(1+x)^2 = 2 \log(1+x)$, deduce the expansion of $\log(1+x)$.

110. In many cases we require only a few terms, three or four, at the beginning of the series which is the expansion of a function; and when the function is composed of constituent functions, of which the expansions are known, then the first three or four terms are readily obtained by a combination of the expansions of the constituents of the function.

Thus

$$e^x \sec x = e^x \div \cos x \\ = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \div \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right)$$

and performing the division as far as five terms of the quotients, using the methods of detached coefficients and contracted division,

$$\begin{array}{r} 1 - \frac{x^2}{2} + \frac{x^4}{24} \dots \bigg) 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \dots \\ \underline{1} \phantom{+ \frac{x^2}{2}} \phantom{+ \frac{x^3}{6}} \phantom{+ \frac{x^4}{24}} \\ 1 + 1 + \frac{1}{6} + 0 \dots \\ \phantom{+ \frac{1}{6}} \\ \phantom{+ \frac{1}{6}} \\ \phantom{+ \frac{1}{6}} \\ \phantom{+ \frac{1}{6}} \end{array}$$

so that $e^x \sec x = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4 + \dots$

Similarly, by the method of detached coefficients in multiplication, the * denoting a missing term,

$$\begin{array}{l} e^x \cos x = 1 + x + * - \frac{1}{3}x^3 - \frac{1}{6}x^4 \dots \\ e^x \sin x = x + x^2 + \frac{1}{3}x^3 + * - \frac{1}{36}x^5 \dots \\ e^x \log(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + * + \frac{3x^5}{40} + \dots \end{array}$$

Examples.

- (1) (i.) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots$
 (ii.) $\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \dots$
 (iii.) $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots$

$$(iv.) \quad \operatorname{cosec} x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \dots$$

$$(v.) \quad e^{\frac{1}{2}x^2} \sin x = x + * - \frac{x^5}{180} - \frac{x^7}{2835} - \frac{x^9}{90720} - \dots$$

$$(vi.) \quad e^{\frac{1}{2}x^2} \cos x = 1 + * - \frac{x^4}{12} - \frac{x^6}{45} - \frac{11x^8}{3360} - \dots$$

$$(vii.) \quad (\cos x)^n \\ = 1 - \frac{nx^2}{2} + \frac{3n^2 - 2n}{24}x^4 - \frac{15n^3 - 30n^2 + 16n}{720}x^6 + \dots$$

$$(viii.) \quad \left(\frac{\sin x}{x}\right)^n \\ = 1 - \frac{nx^2}{6} + \frac{5n^2 - 2n}{360}x^4 - \frac{35n^3 - 42n^2 + 16n}{45360}x^6 + \dots$$

Write down the corresponding expansions of $\tanh x$, $\coth x$, $\operatorname{sech} x$, $\operatorname{cosech} x$,

$$(2) \quad (i.) \quad \tan(\sin x) = x + \frac{x^3}{6} - \frac{x^5}{40} - \frac{107x^7}{5040} \dots$$

$$(ii.) \quad \sin(\tan x) = x + \frac{x^3}{6} - \frac{x^5}{40} - \frac{275x^7}{5040} \dots$$

$$(3) \quad \sqrt[2]{1+x} = e(1 - \frac{1}{2}x + \frac{1}{2} \frac{1}{4}x^2 - \frac{7}{16}x^3 + \frac{2}{5} \frac{4}{7} \frac{4}{8} \frac{7}{6}x^4 - \dots).$$

(This is readily effected by writing

$$\begin{aligned} \sqrt[2]{1+x} &= \exp \log \sqrt[2]{1+x} = \exp(1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \dots) \\ &= e \cdot e^{-\frac{1}{2}x} \cdot e^{\frac{1}{3}x^2} \cdot e^{-\frac{1}{4}x^3} \cdot e^{\frac{1}{5}x^4} \dots, \end{aligned}$$

and then calculating by multiplication the coefficients of x , x^2 , x^3 , x^4 ...)

$$(4) \quad (\tan^{-1}x)/(1+x^2) = x - (1 + \frac{1}{3})x^3 + (1 + \frac{1}{3} + \frac{1}{5})x^5 - \dots \\ + (-1)^{n-1} \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} \right) x^{2n-1} \dots$$

Deduce the expansion of $\frac{1}{2}(\tan^{-1}x)^2$ in powers of x , or of $\frac{1}{2}\theta^2$ in powers of $\tan \theta$.

$$(5) \quad \log \sec x = \frac{1}{2}(\sin x)^2 + \frac{1}{4}(\sin x)^4 + \dots + \frac{1}{2n}(\sin x)^{2n} + \dots$$

*111. *The Exponential Values of the Sine and Cosine.*

Comparing the expansions in § 107 of $\cos x$ and $\cosh x$, $\sin x$ and $\sinh x$, we notice that, if i denotes $\sqrt{(-1)}$, the circular and hyperbolic functions are connected by

$$\cos ix = \cosh x, \sin ix = i \sinh x, \tan ix = i \tanh x;$$

$$\text{so that} \quad \sin(u+iv) = \sin u \cosh v + i \cos u \sinh v, \\ \cos(u+iv) = \cos u \cosh v - i \sin u \sinh v.$$

$$\text{Then} \quad \cos x = \cosh ix = \frac{1}{2}(e^{ix} + e^{-ix}), \\ \sin x = -i \sinh ix = \frac{1}{2i}(e^{ix} - e^{-ix});$$

the *exponential values* of the circular cosine and sine;

$$\text{and} \quad \cosh(v+iu) = \cos u \cosh v + i \sin u \sinh v, \\ \sinh(v+iu) = \cos u \sinh v + i \sin u \cosh v.$$

$$\text{Also} \quad \cos \theta + i \sin \theta = e^{i\theta},$$

$$\text{so that} \quad \cos n\theta + i \sin n\theta = e^{in\theta} = (\cos \theta + i \sin \theta)^n,$$

De Moivre's Theorem, analogous to

$$\cosh u + \sinh u = e^u,$$

$$\text{and} \quad \cosh nu + \sinh nu = e^{nu} = (\cosh u + \sinh u)^n.$$

* 112. *The Resolution of the Trigonometric Functions into Factors.*

A rigorous proof is given in treatises on Trigonometry of the resolution into factors of $\sin \theta$ and $\cos \theta$, in the form

$$\sin \theta = \theta \left(1 - \frac{\theta^2}{\pi^2}\right) \left(1 - \frac{\theta^2}{2^2 \pi^2}\right) \left(1 - \frac{\theta^2}{3^2 \pi^2}\right) \dots \dots \dots (\text{i.})$$

$$\cos \theta = \left(1 - \frac{\theta^2}{\frac{1}{4}\pi^2}\right) \left(1 - \frac{\theta^2}{3^2 \frac{1}{4}\pi^2}\right) \left(1 - \frac{\theta^2}{5^2 \frac{1}{4}\pi^2}\right) \dots \dots \dots (\text{ii.})$$

the factors being originally inferred from the values of θ which make $\sin \theta$ and $\cos \theta$ zero.

Therefore also, changing θ^2 into $-u^2$,

$$\sinh u = u \left(1 + \frac{u^2}{\pi^2}\right) \left(1 + \frac{u^2}{2^2 \pi^2}\right) \left(1 + \frac{u^2}{3^2 \pi^2}\right) \dots \dots \dots (\text{iii.})$$

$$\cosh u = \left(1 + \frac{u^2}{\frac{1}{4}\pi^2}\right) \left(1 + \frac{u^2}{3^2 \frac{1}{4}\pi^2}\right) \left(1 + \frac{u^2}{5^2 \frac{1}{4}\pi^2}\right) \dots \dots \dots (\text{iv.})$$

Again,

$$1 - \cos \theta = 2 \sin^2 \frac{1}{2} \theta \\ = \frac{1}{2} \theta^2 \left(1 - \frac{\frac{1}{4} \theta^2}{\pi^2}\right)^2 \left(1 - \frac{\frac{1}{4} \theta^2}{2^2 \pi^2}\right)^2 \left(1 - \frac{\frac{1}{4} \theta^2}{3^2 \pi^2}\right)^2 \dots \dots \dots \text{(v.)}$$

$$1 + \cos \theta = 2 \cos^2 \frac{1}{2} \theta \\ = 2 \left(1 - \frac{\theta^2}{\pi^2}\right) \left(1 - \frac{\theta^2}{3^2 \pi^2}\right) \left(1 - \frac{\theta^2}{5^2 \pi^2}\right) \dots \dots \dots \text{(vi.)}$$

and

$$1 + \sin \theta = \left(1 + \frac{\theta}{\frac{1}{2} \pi}\right) \left(1 - \frac{\theta}{3 \frac{1}{2} \pi}\right) \left(1 + \frac{\theta}{5 \frac{1}{2} \pi}\right) \dots \dots \dots \text{(vii.)}$$

$$1 - \sin \theta = \left(1 - \frac{\theta}{\frac{1}{2} \pi}\right) \left(1 + \frac{\theta}{3 \frac{1}{2} \pi}\right) \left(1 - \frac{\theta}{5 \frac{1}{2} \pi}\right) \dots \dots \dots \text{(viii.)}$$

so that

$$\frac{1 + \sin \theta}{\cos \theta} = \sec \theta + \tan \theta = \tan \left(\frac{1}{4} \pi + \frac{1}{2} \theta\right) = \sqrt{\frac{1 + \sin \theta}{1 - \sin \theta}} \\ = \frac{\left(1 + \frac{\theta}{\frac{1}{2} \pi}\right) \left(1 - \frac{\theta}{3 \frac{1}{2} \pi}\right) \left(1 + \frac{\theta}{5 \frac{1}{2} \pi}\right) \dots}{\left(1 - \frac{\theta}{\frac{1}{2} \pi}\right) \left(1 + \frac{\theta}{3 \frac{1}{2} \pi}\right) \left(1 - \frac{\theta}{5 \frac{1}{2} \pi}\right) \dots} \dots \dots \dots \text{(ix.)}$$

Taking logarithms of (i.) and expanding, we find

$$\log \frac{\sin x}{x} = \log \left(1 - \frac{x^2}{\pi^2}\right) + \log \left(1 - \frac{x^2}{2^2 \pi^2}\right) + \log \left(1 - \frac{x^2}{3^2 \pi^2}\right) + \dots \\ = -\left(\frac{x}{\pi}\right)^2 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) - \frac{1}{2} \left(\frac{x}{\pi}\right)^4 \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right) \\ - \dots - \frac{1}{n} \left(\frac{x}{\pi}\right)^{2n} S_{2n} - \dots, \dots \dots \dots \text{(x.)}$$

where the series $1^{-p} + 2^{-p} + 3^{-p} + \dots$ or Σn^{-p} is denoted by S_p ; the symbol Σ denoting summation for all successive positive integral values of n from 1 to infinity.

Also, from (ii.), denoting $\Sigma (2n-1)^{-p}$ by T_p ,

$$\log \sec x = \left(\frac{x}{\frac{1}{2} \pi}\right)^2 \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) + \dots + \frac{1}{n} \left(\frac{x}{\frac{1}{2} \pi}\right)^{2n} T_{2n} \dots \text{(xi.)}$$

But $S_p - T_p = 2^{-p} S_p$; so that

$$T_p = (1 - 2^{-p}) S_p, \quad S_p = 2^p T_p / (2^p - 1).$$

Now differentiating (x.) and (xi.) we obtain

$$\cot x = \frac{1}{x} - \frac{2}{\pi} \sum \left(\frac{x}{\pi} \right)^{2n-1} S_{2n} \dots\dots\dots (\text{xii.})$$

$$\tan x = \frac{4}{\pi} \sum \left(\frac{x}{\frac{1}{2}\pi} \right)^{2n-1} T_{2n} \dots\dots\dots (\text{xiii.}).$$

Similarly taking logarithms of equation (ix.) and expanding, gives

$$\begin{aligned} \log(\sec x + \tan x) &= \log \tan\left(\frac{1}{4}\pi + \frac{1}{2}x\right) = \text{gd}^{-1}x \\ &= \frac{2x}{\frac{1}{2}\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right) \\ &+ \frac{2}{3} \left(\frac{x}{\frac{1}{2}\pi}\right)^3 \left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots\right) + \dots \\ &+ \frac{2}{(2n+1)} \left(\frac{x}{\frac{1}{2}\pi}\right)^{2n+1} U_{2n+1} + \dots \dots\dots (\text{xiv.}) \end{aligned}$$

where $1^{-p} - 3^{-p} + 5^{-p} - 7^{-p} + \dots = \Sigma(-1)^{n-1}(2n-1)^{-p}$

is denoted by U_p ; and differentiating

$$\sec x = 1 + \frac{4}{\pi} \sum \left(\frac{x}{\frac{1}{2}\pi} \right)^{2n} U_{2n+1} \dots\dots\dots (\text{xv.});$$

the first term reducing to unity, because (§ 108)

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{1}{4}\pi.$$

The series for $\text{cosec}^2 x$ and $\sec^2 x$ or $\cot^2 x$ and $\tan^2 x$ are obtained by the differentiation of the series (xii.) and (xiii.) for $\cot x$ and $\tan x$.

Also $\frac{\sin(m+n)x}{\sin mx \sin nx}$ and $\frac{\sin(m+n)x}{\cos mx \cos nx}$

can be expanded by writing them in the form

$$\cot mx + \cot nx \quad \text{and} \quad \tan mx + \tan nx.$$

*113. *Bernoulli's Numbers.*

In the general expansion of $\tan x$, $\cot x$, $\operatorname{cosec} x$, $\tanh x$, $\coth x$, $\operatorname{cosech} x$, the coefficients are certain rational numbers called Bernoulli's numbers, which are thus defined.

Suppose $\frac{1}{2}x \frac{e^x + 1}{e^x - 1}$ or $\frac{x}{e^x - 1} + \frac{1}{2}x$

to be expanded in ascending powers of x ; then only *even* powers will occur, because it is an *even* function of x , being unchanged when $-x$ is written for x (§ 50).

Writing the expansion in the form

$$\frac{1}{2}x \frac{e^x + 1}{e^x - 1} = 1 + \frac{x^2}{2!}B_1 - \frac{x^4}{4!}B_2 + \dots + (-1)^{n-1} \frac{x^{2n}}{2n!}B_n + \dots,$$

then $B_1, B_2, \dots, B_n, \dots$ are called *Bernoulli's Numbers*.

With our notation (§ 33)

$$\frac{1}{2}x \frac{e^x + 1}{e^x - 1} = \frac{1}{2}x \coth \frac{1}{2}x,$$

so that, changing $\frac{1}{2}x$ into x ,

$$x \coth x = 1 + \frac{x^2}{2!}2^2B_1 - \frac{x^4}{4!}2^4B_2 + \dots + (-1)^{n-1} \frac{x^{2n}}{2n!}2^{2n}B_n + \dots \text{(i.)}$$

Again, changing x into ix , then (§ 110)

$$ix \coth ix = x \cot x$$

$$= 1 - \frac{x^2}{2!}2^2B_1 - \frac{x^4}{4!}2^4B_2 - \dots - \frac{x^{2n}}{2n!}2^{2n}B_n - \dots$$

$$\text{or } \cot x = \frac{1}{x} - \frac{x}{2!}2^2B_1 - \frac{x^3}{4!}2^4B_2 - \dots - \frac{x^{2n-1}}{2n!}2^{2n}B_n - \dots \text{(ii.)}$$

Now $\tan x = \cot x - 2 \cot 2x$, so that

$$\begin{aligned} \tan x &= \frac{x}{2!}2^2(2^2 - 1)B_1 + \frac{x^3}{4!}2^4(2^4 - 1)B_2 + \dots \\ &\quad + \frac{x^{2n-1}}{2n!}2^{2n}(2^{2n} - 1)B_n + \dots \text{(iii.);} \end{aligned}$$

and therefore changing x into ix ,

$$\tanh x = \frac{x}{2!} 2^2(2^2-1)B_1 - \frac{x^3}{4!} 2^4(2^4-1)B_2 + \dots \\ + (-1)^{n-1} \frac{x^{2n-1}}{2n!} 2^{2n}(2^{2n}-1)B_n + \dots \dots \dots (\text{iv}).$$

Again

$$\operatorname{cosec} x = \tan \frac{1}{2}x + \cot x,$$

so that

$$\operatorname{cosec} x = \frac{1}{x} + \frac{x}{2!} 2B_1 + \frac{x^3}{4!} 2(2^3-1)B_2 + \frac{x^5}{6!} 2(2^5-1)B_3 + \dots (\text{v}),$$

and therefore

$$\operatorname{cosech} x = \frac{1}{x} - \frac{x}{2!} 2B_1 + \frac{x^3}{4!} 2(2^3-1)B_2 - \frac{x^5}{6!} 2(2^5-1)B_3 + \dots (\text{vi}).$$

The first nine numbers of Bernoulli are

$$B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}, B_5 = \frac{5}{66}, B_6 = \frac{691}{2730}, \\ B_7 = \frac{7}{6}, B_8 = \frac{3617}{510}, B_9 = \frac{43867}{798}.$$

Comparing these expansions with those of the last article (§ 112), we notice that

$$S_{2n} = \frac{(2\pi)^{2n}}{2(2n)!} B_n,$$

so that S_{2n}/π^{2n} is a rational number.

If the expansion of $\sec x$ is written

$$\sec x = 1 + \frac{x^2}{2!} E_1 + \frac{x^4}{4!} E_2 + \dots + \frac{x^{2n}}{2n!} E_n + \dots,$$

then $E_1, E_2, \dots, E_n, \dots$ are called Euler's numbers, and

$$E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, \dots$$

Comparing this expansion of $\sec x$ with that of the last article, we find

$$U_{2n+1} = \frac{(\frac{1}{2}\pi)^{2n+1}}{2(2n)!} E_n,$$

so that U_{2n+1}/π^{2n+1} is a rational number.

In these expansions it is useful to employ the abbreviation x_n for $x^n/n!$, and to write down the typical n th term, with Σ before it; and thus Taylor's and Maclaurin's Theorems may be written

$$f(a+x) = fa + \Sigma x_n f^n a, \quad fa = f0 + \Sigma x_n f^n 0.$$

$$\text{Again,} \quad e^x = 1 + \Sigma x_n, \quad a^x = 1 + \Sigma x_n (\log a)^n;$$

$$\cos x = 1 + \Sigma (-1)^n x_{2n}, \quad \cosh x = 1 + \Sigma x_{2n};$$

$$\sin x = \Sigma (-1)^{n-1} x_{2n-1}, \quad \sinh x = \Sigma x_{2n-1};$$

$$\tan x = \Sigma \frac{2^{2n}(2^{2n}-1)}{2n} B_n x_{2n-1},$$

$$\tanh x = \Sigma (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)}{2n} B_n x_{2n-1};$$

$$\cot x = \frac{1}{x} - \Sigma \frac{2^{2n}}{2n} B_n x_{2n-1},$$

$$\coth x = \frac{1}{x} + \Sigma (-1)^{n-1} \frac{2^{2n}}{2n} B_n x_{2n-1};$$

$$\operatorname{cosec} x = \frac{1}{x} + \Sigma \frac{2^{2n-1}-1}{n} B_n x_{2n-1},$$

$$\operatorname{cosech} x = \frac{1}{x} - 2 \Sigma (-1)^{n-1} \frac{2^{2n-1}-1}{n} B_n x_{2n-1};$$

$$\sec x = 1 + \Sigma E_n x_{2n}, \quad \operatorname{sech} x = 1 - \Sigma (-1)^{n-1} E_n x_{2n}.$$

$$\operatorname{vers} x = \Sigma (-1)^{n-1} x_{2n}, \quad \operatorname{versh} x = \Sigma x_{2n}.$$

Since $1 = \cos x \sec x = \sin x \operatorname{cosec} x$, and $\tan x = \sin x \sec x$, the B 's and E 's are connected by relations easily written down; namely, writing E_0 for unity,

$$\frac{E_n}{2n!} - \frac{E_{n-1}}{2!(2n-2)!} + \dots + \frac{(-1)^n E_0}{2n!} = 0,$$

$$\frac{2(2^{2n-1}-1)}{1!2n!} B_n - \frac{2(2^{2n-3}-1)}{3!(2n-2)!} B_{n-1} + \dots + \frac{(-1)^n}{(2n+1)!} = 0,$$

$$\frac{E_{n-1}}{1!(2n-2)!} - \frac{E_{n-2}}{3!(2n-4)!} + \dots + \frac{(-1)^{n-1} E_0}{(2n-1)!} = \frac{2^{2n}(2^{2n}-1)}{2n!} B_n.$$

Suppose $\theta = \text{gd } u$, or $u = \text{gd}^{-1}\theta = \log(\sec \theta + \tan \theta)$;
 then $\cos \theta \cosh u = 1$, or $\cosh i\theta \cos iu = 1$,
 so that $iu = \text{gd } i\theta$, or $i\theta = \log(\sec iu + \tan iu)$.

Now $u = \log(\sec \theta + \tan \theta) = \int_0^\theta (\sec \theta) d\theta = \theta + \sum E_n \theta_{2n+1}$,
 so that, conversely, $\theta = u - \sum (-1)^{n-1} E_n u_{2n+1}$,
 a curious instance of the *reversion of a series*.

We may compare the reversions

$y = e^x - 1 = \sum x_n$, and $x = \log(1 + y) = \sum (-1)^{n-1} y^n / n$;
 also the reversions of $y = (1 + x)^m$, $\sin x$, ...

Examples.

(1) Calculate to seven decimals from these series the value
 of $\tan 18^\circ = \tan \frac{1}{10}\pi$; also of $\sin 18^\circ$ and $\cos 18^\circ$.

(2) Prove that $\text{cosec } 1^\circ = 57.3$, $\cot 1^\circ = 57.29$;

$$\text{cosec } 1' = 3437.7 = \cot 1' ;$$

$$\text{cosec } 1'' = 206265 = \cot 1'' ;$$

$$\text{cosec } 8''.76 = 23546 ;$$

and thence, taking the sun's parallax as $8''.76$,
 prove that the distance of the sun is about 150
 million kilometers, or 81 million nautical miles.

(3) In the expansion $(1-x)^{-\frac{1}{2}} \exp(\frac{1}{2}x + \frac{1}{4}x^2) = \sum A_n x_n$, prove
 that A_n is always an integer.

(4) Given $\exp(e^x - 1) = 1 + \sum L_n x_n$, prove that, for

$$n = 1, 2, 3, \dots, L_n = 1, 2, 5, 15, 52, \dots$$

(5) Given $\sin \log(1+x) = \sum A_n x_n$, $\cos \log(1+x) = 1 + \sum B_n x_n$,
 calculate the first eight A 's and B 's.

(6) Prove that the coefficient of x^{2n} in the expansion of
 $x/(e^x - 1)$ is $2S_{2n}/(2\pi)^{2n}$; and that

$$\log \frac{\cosh x - \cos x}{x^2} + \sum \frac{x^{2n}}{2n!} \frac{2^n B_n \cos \frac{1}{2} n \pi}{n} = 0.$$

(7) Prove that $S_3 = 1.2020569$; and that

$$\sum n^{-2} (n+1)^{-2} = \frac{1}{3} \pi^2 - 3 ; \quad \sum n^{-3} (n+1)^{-3} = 10 - \pi^2 ;$$

$$\sum n^{-4} (n+1)^{-4} = \frac{1}{4} \pi^4 + \frac{1}{3} \pi^2 - 35.$$

114. *The Remainder in Taylor's Series.*

The previous expansions extend to an infinite number of terms, and are therefore only true when *convergent*.

But some functions, for instance $\sec^{-1}x$, $\cosh^{-1}x$, or $\coth^{-1}x$, cannot be expanded in an infinite series in ascending powers of x , because x must be greater than unity, and the expansion by Taylor's or Maclaurin's Theorem would be *divergent*, and the theorem is then said to fail.

This difficulty will be avoided if we can make the series terminate after a finite number of terms; we shall proceed to explain how this can be done.

Suppose $f(a+h)$ expanded, in Taylor's Series,

$$f(a+h) = fa + hf'a + \frac{h^2}{2!}f''a + \dots + \frac{h^n}{n!}f^n a + R,$$

where
$$R = \frac{h^{n+1}}{(n+1)!}f^{n+1}a + \frac{h^{n+2}}{(n+2)!}f^{n+2}a + \dots$$

Since all the terms of R involve $\frac{h^{n+1}}{(n+1)!}$ as a factor, we

may put
$$R = \frac{h^{n+1}}{(n+1)!}P,$$

and seek to determine an expression for P .

We shall prove that $P = f^{n+1}(a + \theta h)$, where θ is a proper fraction, some unknown function of a and h ; then

$$R = \frac{h^{n+1}}{(n+1)!}f^{n+1}(a + \theta h),$$

and R is called *Lagrange's Form of the Remainder* in Taylor's Series; so that Taylor's Theorem is now

$$f(a+h) = fa + hf'a + \frac{h^2}{2!}f''a + \dots + \frac{h^n}{n!}f^n a + \frac{h^{n+1}}{(n+1)!}f^{n+1}(a + \theta h),$$

thus avoiding the use of an infinite series; and incidentally establishing Taylor's Theorem in a rigorous manner.

To prove that $P = f^{n+1}(a + \theta h)$,
we write down a function Fx , such that

$$Fx = fx + (a + h - x)f'x + \frac{(a + h - x)^2}{2!}f''x + \dots \\ + \frac{(a + h - x)^n}{n!}f^nx + \frac{(a + h - x)^{n+1}}{(n + 1)!}P;$$

then $F'x = \frac{(a + h - x)^n}{n!}(f^{n+1}x - P);$

also $Fa = f(a + h)$ and $F(a + h) = f(a + h).$

If we draw a curve BQK , whose equation is $y = Fx$, with ordinates AB, MQ, HK , at B, Q, K , then if

$$OA = a, \quad AB = Fa = f(a + h);$$

and if $OH = a + h, HK = F(a + h) = f(a + h);$

and the chord BK is therefore parallel to the axis of x .

Now if $fx, f'x, f''x, \dots f^nx$ are all finite and change gradually between $x = a$ and $x = a + h$, then Fx and $F'x$ are also finite and continuous; and therefore at some point Q of the curve BQK between B and K the tangent is parallel to the axis of x .

Exceptional cases where Fx is not continuous are seen represented in fig. 31.

If $a + \theta h$ is the abscissa of this point Q , θ is a proper fraction, and $AM = \theta h$; then $F'(a + \theta h) = 0$,

or
$$\frac{(h - \theta h)^n}{n!} \{f^{n+1}(a + \theta h) - P\} = 0,$$

and therefore $P = f^{n+1}(a + \theta h).$

*115. The actual value of θ is seldom assignable, and is not a matter of practical interest.

We can, however, assign an approximate value, and a few terms of a series in powers of h , giving a closer approximate value.

For expanding R in two series, writing h_n for $h^n/n!$,

$$R = h_{n+1}(f^{n+1}a + \theta h f^{n+2}a + \theta^2 h_2 f^{n+3}a + \dots)$$

$$R = h_{n+1}f^{n+1}a + h_{n+2}f^{n+2}a + h_{n+3}f^{n+3}a + \dots;$$

an equation for determining θ by the method of successive approximation or reversion of series (§ 85).

Thus, as a first approximation, $\theta = 1/(n+2)$; and to a second approximation,

$$\theta = \frac{1}{n+2} + \frac{n+1}{2(n+2)^2(n+3)} \frac{f^{n+3}a}{f^{n+2}a} h;$$

and so on; the next term being

$$\frac{(n+1)(5n+12)f^{n+2}af^{n+4}a - 3(n+1)(n+4)(f^{n+3}a)^2}{6(n+2)^3(n+3)(n+4)(f^{n+2}a)^2} h^2.$$

For instance, if $n=0$, or as in § 7,

$$f(a+h) = fa + hf'(a + \theta h),$$

$$\text{then} \quad \theta = \frac{1}{2} + \frac{f'''a}{f''a} \frac{h}{24} + \frac{f''af''''a - (f'''a)^2}{(f''a)^2} \frac{h^2}{48} + \dots;$$

$$\text{and if} \quad f(a+h) = fa + hf'a + h_2f''(a + \theta h),$$

$$\text{then} \quad \theta = \frac{1}{3} + \frac{f^4a}{f^3a} \frac{h}{36} + \frac{17f^3af^5a - 15(f^4a)^2}{(f^3a)^2} \frac{h^2}{1620} + \dots$$

*116. If we had put $R = hP$, and

$$Fx = fx + (a+h-x)f'x + \frac{(a+h-x)^2}{2!}f''x + \dots$$

$$+ \frac{(a+h-x)^n}{n!}f^n x + (a+h-x)P,$$

$$\text{so that} \quad Fa = F(a+h) = f(a+h);$$

$$\text{then} \quad F'x = \frac{(a+h-x)^n}{n!}f^{n+1}x - P,$$

$$\text{and} \quad F'(a+\theta h) = 0, \text{ when } P = \frac{h^n}{n!}(1-\theta)^nf^{n+1}(a+\theta h);$$

$$\text{and then} \quad R = \frac{h^{n+1}}{n!}(1-\theta)^nf^{n+1}(a+\theta h),$$

Cauchy's Form of the Remainder in Taylor's Series.

More generally, if we had put $R = h^{p+1}P$, and

$$Fx = fx + (a+h-x)f'x + (a+h-x)_2f''x + \dots \\ + (a+h-x)_nf^nx + (a+h-x)^{p+1}P,$$

so that

$$Fa = F(a+h) = f(a+h),$$

then
$$F'x = \frac{(a+h-x)^n}{n!} f^{n+1}x - (p+1)(a+h-x)^p P;$$

and

$$F'(a+\theta h) = 0,$$

when
$$P = \frac{h^{n-p}}{n!(p+1)} (1-\theta)^{n-p} f^{n+1}(a+\theta h),$$

and then
$$R = \frac{h^{n+1}}{n!(p+1)} (1-\theta)^{n-p} f^{n+1}(a+\theta h);$$

Schlömilch and Roche's Form of the Remainder.

When $p = n$, this becomes Lagrange's Remainder, and Cauchy's when $p = 0$.

117. Put $a = 0$, and change h into x ; then

$$fx = f0 + xf'0 + x_2f''0 + \dots + x_nf^n0 + x_{n+1}f^{n+1}(\theta x),$$

Maclaurin's Theorem with Lagrange's Remainder; so that now we may write $P = f^{n+1}(\theta x)$.

Thus P has the following values for the corresponding functions, when expanded in powers of x .

(i.) $(a+x)^m, \quad P = m(m-1)\dots(m-n)(a+\theta x)^{m-n-1};$

(ii.) $\sin(x+a), \quad P = \sin\{\theta x + a + (n+1)\frac{1}{2}\pi\}.$

(iii.) $\sinh x$, or $\cosh x, \quad P = \frac{1}{2}\{e^{\theta x} \pm (-1)^ne^{-\theta x}\};$

(iv.) e^x , or $a^x, \quad P = e^{\theta x}$, or $a^{\theta x}(\log a)^{n+1};$

(v.) $\log(1+x), \quad P = (-1)^nn!(1+\theta x)^{-n-1};$

(vi.) $\tanh^{-1}x, \quad P = n! \frac{(1+\theta x)^{n+1} - (-1+\theta x)^{n+1}}{2(1-\theta^2x^2)^{n+1}};$

(vii.) $\tan^{-1}x,$

$$P = n!(1+\theta^2x^2)^{-\frac{1}{2}(n+1)} \sin\{(n+1)(\frac{1}{2}\pi + \tan^{-1}\theta x)\};$$

(viii.) $\exp(x \cos \alpha) \cos(x \sin \alpha),$

$$P = \exp(\theta x \cos \alpha) \cos\{\theta x \sin \alpha + (n+1)\alpha\}.$$

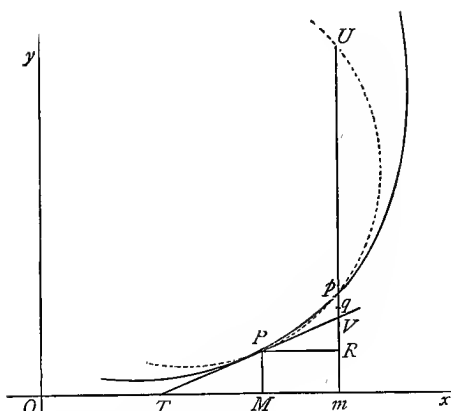


Fig. 42

118. *Geometrical Illustration of Taylor's Theorem. Contact of Different Orders.*

If $y = fx$ is the equation of a curve Pp , and if $OM = x$, then $MP = fx$ (fig. 42).

If $Mm = h$, then $Om = x + h$, and $mp = f(x + h)$.

Draw the tangent TPV at P , cutting mp in V , and draw PR parallel to Ox .

Then $\tan RPV = f'x$,

so that $RV = hf'x$, and $mV = fx + hf'x$.

But, by Taylor's Theorem, as far as three terms,

$$f(x + h) = fx + hf'x + \frac{1}{2}h^2f''(x + \theta h);$$

so that $Vp = \frac{1}{2}h^2f''(x + \theta h)$.

Describe a circle touching the curve at P and passing through p , and let Vp produced meet the circle in U .

Then, since $PV^2 = VU \cdot Vp$,

therefore $VU = PV^2 / Vp = (PR^2 + RV^2) / Vp$

$$= \frac{h^2\{1 + (f'x)^2\}}{\frac{1}{2}h^2f''(x + \theta h)} = \frac{1 + (f'x)^2}{\frac{1}{2}f''(x + \theta h)}.$$

Now make p approach to coincidence with P by diminishing h to zero; the circle becomes the circle of curvature at P , and VU becomes the chord of curvature parallel to the axis of y , and as before (§ 94),

$$= \frac{1 + (f'x)^2}{\frac{1}{2}f''x} = 2 \left(1 + \frac{dy^2}{dx^2} \right) / \frac{d^2y}{dx^2}.$$

Thus, at a maximum or minimum y , $dy/dx=0$, and $d^2y/dx^2=1/\rho$, the curvature, estimated positive *upwards*.

Suppose $x=a$ makes $dy/dx=0$, then $f'a=0$; and

$$f(a \pm h) = fa + \frac{1}{2}h^2f''(a \pm \theta h).$$

When h is small $f''a$ may be written for $f''(a \pm \theta h)$; and, when $f'a=0$, the ordinate fa is less than the adjacent ordinates $f(a \pm h)$ if $f''a$ is *positive*, and fa has therefore a *minimum* value; but if $f''a$ is *negative*, then fa is greater than $f(a \pm h)$, and fa is therefore a *maximum*; the Theory of Maximum and Minimum is often discussed in this manner by the aid of Taylor's Theorem.

Let $f(x+h) = fx + hf'x + h_2f''x + h_3f'''(x + \theta h)$, when expanded by Taylor's Theorem as far as four terms; and let Pq be the arc of the parabola, which has its axis parallel to the axis of y , and which *osculates* the curve Pp at P , that is, which has the same circle of curvature at P ; then $Vq = \frac{1}{2}h^2f''x$, so that $qp = \frac{1}{6}h^3f'''(x + \theta h)$.

The geometrical interpretation of

$$f(x+h) = fx + hf'(x + \theta h)$$

has been given in Chapter I., § 7.

The graph of the equation

$$y = fx + hf'x + h_2f''x + h_3f'''x$$

will for different values of h , keeping x constant, be a curve of the third degree, touching the curve Pp at P , and having the same curvature, and in addition the same value of d^3y/dx^3 at P ; it is then said to have a *contact of the third order* with the curve $y = fx$.

Generally, the graph of the equation

$$y = fx + hf'x + h_2f''x + \dots + h_nf^n x$$

will represent for variable values of h a curve of the n^{th} degree, and it is said to have *contact of the n^{th} order* at P with $y = fx$; two curves being said to have a contact of the n^{th} order at a point of intersection when the first n d.c.'s of y with respect to x (and therefore of x with respect to y) are equal in the two curves at this point.

*119. *Infinitesimals.*

In discussing the properties of a curve OP in the neighbourhood of a point O , it is convenient to take O as origin, and the tangent Ox and normal Oy as coordinate axes, as in § 96; but now to take the arc s measured from O as the independent variable. The student can easily supply the figure.

Then by Taylor's or Maclaurin's Theorem,

$$x = x's + x''s_2 + x'''s_3 + \dots, \quad y = y's + y''s_2 + y'''s_3 + \dots;$$

the accents denoting differentiation with respect to s and that s is afterwards made zero, so as to refer to O .

$$\text{Now} \quad \frac{dx}{ds} = \cos \psi, \quad \frac{dy}{ds} = \sin \psi;$$

$$\frac{d^2x}{ds^2} = -\sin \psi \frac{d\psi}{ds}, \quad \frac{d^2y}{ds^2} = \cos \psi \frac{d\psi}{ds};$$

$$\frac{d^3x}{ds^3} = -\cos \psi \frac{d^2\psi}{ds^2} - \sin \psi \frac{d^2\psi}{ds^2},$$

$$\frac{d^3y}{ds^3} = -\sin \psi \frac{d^2\psi}{ds^2} + \cos \psi \frac{d^2\psi}{ds^2};$$

$$\frac{d^4x}{ds^4} = \sin \psi \left(\frac{d^3\psi}{ds^3} - \frac{d^3\psi}{ds^3} \right) - 3 \cos \psi \frac{d\psi}{ds} \frac{d^2\psi}{ds^2},$$

$$\frac{d^4y}{ds^4} = -3 \sin \psi \frac{d\psi}{ds} \frac{d^2\psi}{ds^2} - \cos \psi \left(\frac{d^3\psi}{ds^3} - \frac{d^3\psi}{ds^3} \right); \dots$$

Denoting by c the curvature ψ' or $1/\rho$ at O , then, by putting $s=0$ and $\psi=0$, we find

$$\begin{aligned}x' &= 1, y' = 0; x'' = 0, y'' = c; x''' = -c^2, y''' = c'; \\x''' &= -3cc', y''' = -c^3 + c'; \dots\end{aligned}$$

so that, to three terms of the expansion,

$$x = s - c^2 s_3 - 3cc' s_4 + \dots, y = cs_2 + c' s_3 - (c^3 - c'') s_4 \dots;$$

or, since $c = \frac{1}{\rho}, c' = -\frac{\rho'}{\rho^2}, c'' = \frac{2\rho'^2}{\rho^3} - \frac{\rho''}{\rho^2}, \dots,$

$$x = s - \frac{s^3}{6\rho^2} + \frac{s^4\rho'}{8\rho^3} \dots, y = \frac{s^2}{2\rho} - \frac{s^3\rho'}{6\rho^2} - \frac{s^4}{24\rho^3}(1 - 2\rho'^2 + \rho\rho'') \dots$$

Then $\cos \psi = \frac{dx}{ds} = 1 - \frac{s^2}{2\rho^2} + \frac{s^3\rho'}{2\rho^3} \dots,$

$$\sin \psi = \frac{dy}{ds} = \frac{s}{\rho} - \frac{s^2\rho'}{2\rho^2} - \frac{s^3}{6\rho^3}(1 - 2\rho'^2 + \rho\rho'') \dots,$$

so that

$$\tan \psi = \sin \psi / \cos \psi = \frac{s}{\rho} - \frac{s^2\rho'}{2\rho^2} + \frac{s^3}{3\rho^3}(1 + \rho'^2 - \frac{1}{2}\rho\rho'') \dots$$

Along Op , the circle of curvature at O , we take c or ρ as constant, so that $c', \rho', c'', \rho'', \dots$ vanish; and then

$$x = s - \frac{s^3}{6\rho^2} \dots, y = \frac{s^2}{2\rho} - \frac{s^4}{24\rho^3} \dots;$$

and therefore, if the arcs OP and Op are each equal to s , we find that $\text{lt } Pp/s^3 = \frac{1}{6}c^3\rho'$

Let the normal at P meet the normal at O in Q , so that Q is ultimately the centre of curvature at O ; then

$$OQ = y + x \cot \psi = \rho + \frac{1}{2}s \frac{d\rho}{ds} + \frac{s^2}{6\rho}(\rho\rho'' - \rho'^2) \dots;$$

and $\text{lt}(OQ - \rho)/s = \frac{1}{2}d\rho/ds.$

Similarly, if R denotes the radius of the circle which touches OP at O and passes through P ,

$$2R = x^2/y + y = 2\rho + \frac{2}{3}s\rho' \dots,$$

so that $\text{lt}(R - \rho)/s = \frac{1}{3}d\rho/ds.$

Any magnitude o on the figure is called an *infinitesimal of the n^{th} order*, with respect to s , if $\text{lt } o/s^n$ is a finite quantity; thus x and ψ are infinitesimals of the first order, y of the second order, Pp of the third order,

In d^ny/dx^n , d^ny is of the n^{th} order compared with dx .

As exercises in Infinitesimals, the student may prove that, if the chord PNP' is parallel to the tangent at O .

- (i.) $\text{lt} (OT - PT) / s^2 = \frac{1}{2} c' / c$;
- (ii.) $\text{lt} (PQ - OQ) / s^3 = \frac{1}{12} c'$;
- (iii.) $\text{lt} (TQ - OQ) / s^2 = \frac{1}{8} c$;
- (iv.) $\text{lt} (OT + TP - \text{arc } OP) / s^3 = \frac{1}{12} c^2$;
- (v.) $\text{lt} (\text{arc } OP - \text{chord } OP) / s^3 = \frac{1}{24} c^2$;
- (vi.) $\text{lt} (PG - PM) / s^4 = \frac{1}{4} c^3$;
- (vii.) $\text{lt} (\text{arc } OP - \text{arc } OP') / s^2 = \frac{1}{3} c' / c$;
- (viii.) $\text{lt} (NP - NP') / s^2 = \frac{1}{3} c' / c$.

The equation of the conic which touches the curve OP at O is of the form $y = \frac{1}{2}(ax^2 + 2hxy + by^2)$; and if in addition the conic osculates the curve at O , that is has the same curvature, then $a = c = 1/\rho$, the curvature (§ 91).

We can make the values of y''' in the curve and the conic the same, by taking $h = \frac{1}{3} c' / c$, and now the conic has a *contact of the third order*, b being still arbitrary; and the locus of the centre of the conic is the straight line $ax + hy = 0$.

If in addition we determine b so that y'''' is the same in the curve and the conic, then the conic has a contact of the *fourth order* with the curve, and it is called the *conic of closest contact*; it may be considered the ultimate form of the conic which passes through five consecutive points near O ; and we shall find

$$b = \frac{1}{3} \frac{c''}{c^2} - \frac{4}{9} \frac{c'^2}{c^3} + c.$$

120. *Indeterminate Forms, or Singular Values.*

In general the value of a function of x is determinate, and is obtained by substitution for any particular value of the independent variable x .

But when for a particular value of x , say $x=a$, the function assumes one of the *indeterminate forms*, or *singular values*, $0/0$, ∞/∞ , $\infty \times 0$, $\infty - \infty$, 1^∞ , ∞^0 , 0^0 , the real value must be obtained by the *method of limits*; that is, the value of the function must be found for $x=a+h$, where h is small, and reduced and simplified as much as possible by cancelling factors, etc., and afterwards h must be made to vanish.

We have seen this exemplified in Chapter I. in finding the derivatives of the simple functions; for

$$\frac{dfx}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - fx}{h}$$

assumes at first the indeterminate form $0/0$, before reduction; and in the evaluation of these indeterminate forms we have employed lemmas which enable us to write down the values, when $x=0$, of $(\sin mx)/nx$, $\{(\tan mx)/nx\}^p$, $(\sin^{-1}mx)/nx$, $\{(\tan^{-1}mx)/nx\}^p$, $\sqrt[n]{1+x}$, $(1+x/a)^{b/x}$, $\{\log_a(1+x)\}/x$, etc.

By ordinary algebraical and trigonometrical reductions the indeterminate form may in general be evaluated; but the Differential Calculus affords a general method.

Thus, suppose that when $x=a$, the function fx/Fx assumes the indeterminate form $0/0$, because $fa=0$ and $Fa=0$. Then, when $x=a$,

$$\begin{aligned} \frac{fx}{Fx} &= \lim_{h \rightarrow 0} \frac{f(a+h)}{F(a+h)} = \lim_{h \rightarrow 0} \frac{fa + hf'a + h_2 f''a + \dots}{Fa + hF'a + h_2 F''a + \dots} \\ &= \lim_{h \rightarrow 0} \frac{f'a + \frac{1}{2}hf''a + \dots}{F'a + \frac{1}{2}hF''a + \dots} = \frac{f'a}{F'a}. \end{aligned}$$

If however $f'a=0$ and $F'a=0$, the true value is $\frac{f''a}{F''a}$; and so on, till the value is obtained.

Thus, when $x=a$,

$$\frac{x^n - a^n}{x - a} = \frac{0}{0} = \text{lt} \frac{nx^{n-1}}{1} = na^{n-1}.$$

Or, putting $x=a+h$, as in § 4,

$$\text{lt} \frac{x^n - a^n}{x - a} = \text{lt} \frac{(a+h)^n - a^n}{h} = na^{n-1}.$$

If n is a positive integer, then $x-a$ divides $x^n - a^n$, and the quotient,

$$x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1},$$

becomes na^{n-1} , when $x=a$.

Again, as another example, when $x=0$,

$$\begin{aligned} \frac{\sinh x - \sin x}{x^3} &= \frac{0}{0}, = \text{lt} \frac{\cosh x - \cos x}{3x^2} = \frac{0}{0}, \\ &= \text{lt} \frac{\sinh x + \sin x}{6x} = \frac{0}{0}, = \text{lt} \frac{\cosh x + \cos x}{6} = \frac{1}{3}; \end{aligned}$$

and thus $\sinh x - \sin x$ is an infinitesimal of the third order with respect to x .

But by using the expansions of $\sin x$ and $\sinh x$, then this function can be evaluated as follows:—when $x=0$,

$$\begin{aligned} &(\sinh x - \sin x)/x^3 \\ &= \text{lt}(x + x_3 + x_5 + x_7 + \dots - x + x_3 - x_5 + x_7 - \dots)/x^3 \\ &= 2 \text{lt}(x_3 + x_7 + x_{11} + \dots)/x^3 \\ &= 2 \text{lt}\left(\frac{1}{3!} + \frac{x^4}{7!} + \frac{x^8}{11!} + \dots\right) = \frac{2}{3!} = \frac{1}{3}, \text{ as before.} \end{aligned}$$

121. The indeterminate form ∞/∞ can be thrown into the form $0/0$ by interchanging numerator and denominator, and is evaluated in the same manner.

For instance, when $x = \frac{1}{2}\pi$, $\frac{\sec(2n+1)x}{\sec(2m+1)x} = \frac{\infty}{\infty}$;

but, when numerator and denominator are interchanged,

$$\begin{aligned} &= \frac{\cos(2m+1)x}{\cos(2n+1)x} = \frac{0}{0} = \text{lt} \frac{(2m+1)\sin(2m+1)x}{(2n+1)\sin(2n+1)x} \\ &= \frac{2m+1}{2n+1}(-1)^{m+n}. \end{aligned}$$

The indeterminate form $\infty \times 0$ assumes the form $0/0$ by throwing the ∞ into the denominator. For instance,

when $x = \infty$, $2^x \sin \frac{a}{2^x} = \infty \times 0 = a \frac{\sin a/2^x}{a/2^x} = \frac{0}{0} = a$ (§ 16).

The indeterminate form $\infty - \infty$ can also be made to assume the form $0/0$ by reduction of the terms to a single fraction. Thus, for example,

when $x = \frac{1}{2}\pi$, $\sec x - \tan x = \infty - \infty$

$$= \frac{1 - \sin x}{\cos x} = \frac{0}{0} = \frac{1 - \sin x}{\sqrt{(1 - \sin^2 x)}} = \sqrt{\frac{1 - \sin x}{1 + \sin x}} = 0;$$

and, again, $\frac{1}{2}\pi \sec x - x \tan x = \infty - \infty$

$$= \frac{\frac{1}{2}\pi - x \sin x}{\cos x} = \frac{0}{0} = \text{lt} \frac{-\sin x - x \cos x}{-\sin x} = 1.$$

122. To evaluate 1^∞ , ∞^0 , 0^0 , take the logarithm; this will be found to assume the form $0/0$ or ∞/∞ , and can then be evaluated by the preceding rules.

For instance, when $x = 0$, $(\cos mx)^{n/x^2} = 1^\infty$; and

$$\begin{aligned} \log(\cos mx)^{n/x^2} &= n \frac{\log \cos mx}{x^2} = \frac{0}{0} \\ &= n \text{lt} \frac{-m \tan mx}{2x} = \frac{0}{0} \\ &= n \text{lt} \frac{-m^2 \sec^2 mx}{2} = -\frac{1}{2} m^2 n, \end{aligned}$$

and therefore $(\cos mx)^{n/x^2} = \exp(-\frac{1}{2} m^2 n)$.

Examples.—Prove that, when

$$(1) \ x=2, \quad \frac{x^3-19x+30}{x^3-2x^2-9x+18}=\frac{7}{5};$$

$$x=3, \quad \dots\dots\dots=\frac{4}{3};$$

$$(2) \ x=a, \quad \frac{\sqrt{a}-\sqrt{x}+\sqrt{(a-x)}}{\sqrt{(a^2-x^2)}}=\frac{1}{\sqrt{(2a)}}.$$

$$(3) \ x=0, \quad \frac{\tan x + \sec x - 1}{\tan x - \sec x + 1} = 1.$$

$$(4) \ x=\frac{1}{2}\pi, \quad \frac{\cos x + 1 - \sin x}{\cos x - 1 + \sin x} = 1.$$

$$(5) \ x=\frac{1}{4}\pi, \quad \frac{\sin x - \cos x}{\sin 2x - \cos 2x - 1} = \frac{1}{2}\sqrt{2}.$$

$$(6) \ x=\frac{1}{6}\pi, \quad \frac{1 - \sin x - 2 \sin^2 x}{1 - 3 \sin x + 2 \sin^2 x} = 3.$$

$$(7) \ x=0, \quad \frac{(2 \sin x - \sin 2x)^2}{(\sec x - \cos 2x)^3} = 0.064.$$

$$(8) \ x=1, \quad \frac{\log x}{x-1} = 1.$$

$$(9) \ x=0, \quad \frac{\sinh x}{\log(1+x)} = 1.$$

$$(10) \ n=-1, \quad \frac{x^n - a^n}{n+1} = \log \frac{x}{a}.$$

(In this way $\int_a^x x^{-1} dx$ is deduced from $\int_a^x x^n dx$.)

$$(11) \ x=a, \quad \frac{x^x - a^x}{x^a - a^a} = 1.$$

$$(12) \ x=0, \quad \sqrt{x}/(1+x) = e, \quad \{\sqrt{x}/(1+x) - e\}/x = -\frac{1}{2}e, \\ \{\sqrt{x}/(1+x) - e + \frac{1}{2}ex\}/x^2 = \frac{1}{2}\frac{1}{4}e, \\ \{\sqrt{x}/(1+x) - e + \frac{1}{2}ex - \frac{1}{2}\frac{1}{4}ex^2\}/x^3 = -\frac{7}{16}e, \dots$$

$$\begin{aligned}
 (13) \quad & x=0, (\text{vers } x)/x^2 = \frac{1}{2}, \\
 & (\sin x - x \cos x)/x^3 = \frac{1}{3}, \\
 & (x - \sin x)/x^3 = \frac{1}{6}, \\
 & (\sin^{-1}x - x)/x^3 = \frac{1}{6}, \\
 & (1 - \cos x - \frac{1}{2}x \sin x)/x^4 = \frac{1}{24}, \\
 & (\sin x + \sinh x - 2x)/x^5 = \frac{1}{60}, \\
 & (\cosh x - \cos x - x^2)/x^6 = \frac{1}{360}.
 \end{aligned}$$

$$(14) \quad x=0, \{\tan(\sin x) - \sin(\tan x)\}/x^7 = \frac{1}{30}.$$

$$(15) \quad x=0, \frac{x - \sin x}{\tan x - x} = \frac{1}{2},$$

$$\frac{\sin^{-1}x - \sin x}{\tan x - \tan^{-1}x} = \frac{1}{2}.$$

$$\begin{aligned}
 (16) \quad x=a, \quad & \frac{fx}{Fx} = \frac{fa}{Fa}; \\
 & \frac{fx - fa}{Fx - Fa} = \frac{f'a}{F'a}; \\
 & \frac{fx - fa - (x-a)f'a}{Fx - Fa - (x-a)F'a} = \frac{f''a}{F''a}; \\
 & \frac{fx - fa - (x-a)f'a - \frac{1}{2}(x-a)^2f''a}{Fx - Fa - (x-a)F'a - \frac{1}{2}(x-a)^2F''a} = \frac{f'''a}{F'''a};
 \end{aligned}$$

and so on.

$$(17) \quad x=\infty, \frac{ax+b}{Ax+B} = \frac{a}{A};$$

$$\frac{ax^2+2bx+c}{Ax^2+2Bx+C} = \frac{a}{A};$$

$$\frac{ax^m+bx^{m-1}+cx^{m-2}+\dots+k}{Ax^n+Bx^{n-1}+Cx^{n-2}+\dots+K} = 0, \frac{a}{A}, \text{ or } \infty,$$

according as m is $<$, $=$, or $> n$.

Write down the values when $x=0$.

$$(18) \quad x=0, \frac{\log \cot x}{\operatorname{cosec} x}=0.$$

$$(19) \quad x=0, x^n \log x = 0 \text{ or } -\infty, \\ \text{according as } n \text{ is positive or negative.}$$

$$(20) \quad x=a, (a-x) \tan \frac{1}{2} \pi x/a = 2a/\pi.$$

$$(21) \quad x=\infty, x(\sqrt[n]{a}-1)=\log a.$$

$$(22) \quad x=\infty, \sqrt[n]{x^n+a^n}-x=0.$$

$$(23) \quad x=1, \frac{x}{\log x} - \frac{1}{\log x} = 1.$$

$$(24) \quad x=0, \frac{1}{x} - \cot x = 0; \frac{1}{x^2} - \cot^2 x = \frac{2}{3}, \operatorname{cosec}^2 x - \frac{1}{x^2} = \frac{1}{3}.$$

$$(25) \quad x=1, x^{1/(1-x)} = 1/e.$$

$$(26) \quad x=\infty, (1+1/x)^x = e.$$

$$(27) \quad n=\infty,$$

$$(\cos a/n)^n = 1; (\cos a/n)^{n^2} = \exp(-\frac{1}{2}a^2); (\cos a/n)^{n^3} = 0;$$

$$\left(\frac{\sin a/n}{a/n}\right)^n = 1, \left(\frac{\sin a/n}{a/n}\right)^{n^2} = \exp(-\frac{1}{6}a^2); \left(\frac{\sin a/n}{a/n}\right)^{n^3} = 0.$$

$$(28) \quad x=0, \left(\frac{\sin x}{x}\right)^{(\operatorname{cosec} x)^r} = 1, \exp(-\frac{1}{6}), \text{ or } 0;$$

$$(\sec ax)^{(\cot bx)^r} = 1, \exp \frac{1}{2} a^2/b^2, \text{ or } \infty;$$

according as r is $<$, $=$, or > 2 .

$$(29) \quad x=0, (\cos x)^{\cot x} = 1.$$

$$(30) \quad x=\frac{1}{2}\pi, (\sin x)^{\tan x} = 1; (\sin x)^{\tan^2 x} = 1/\sqrt{e}.$$

$$(31) \quad x=\frac{1}{4}\pi, (\tan x)^{\tan 2x} = 1/e.$$

$$(32) \quad x=0, (1+1/x)^x = 1; (\cot x)^{\sin x} = 1.$$

$$(33) \quad x=\infty, \sqrt[n]{x} = 1, \sqrt[n]{1+x} = 1.$$

$$(34) \quad x=0, x^x = 1; \sqrt[n]{x} = 0; (\sin x)^{\tan x} = 1.$$

CHAPTER V.

PARTIAL DIFFERENTIATION AND INTEGRATION.

123. *Functions of Two Independent Variables.*

When y is a function fx of a single variable x , the relation $y = fx$ is exhibited graphically by means of a plane curve, in which the abscissa is x and the corresponding ordinate y is fx (fig. 1), the curve being called the graph of the function fx (§ 5).

But when a variable quantity z is a function of two independent variables x and y , expressed by the notation

$$z = f(x, y) \dots\dots\dots (A),$$

then x and y may be supposed to be the coordinates of any point on a datum (horizontal) plane, and z to be the height of a surface above this point on the datum plane, *e.g.* the surface of the land; so that the relation (A) is the equation of the surface, and the graph of a function $f(x, y)$ of two independent variables x and y is a surface.

For instance, the relation $pv = R\theta$, connecting p the pressure, v the volume, and θ the absolute temperature of a given quantity of a perfect gas, where R is a constant, may be represented graphically by means of the surface,

$$cz = xy,$$

where x represents the volume v , y the pressure p , and z the temperature θ .

With coordinate axes Ox , Oy , Oz , at right angles, the curve $y = fx$ will, by revolution round Ox , sweep out the surface whose equation is

$$\sqrt{(y^2 + z^2)} = fx, \text{ or } y^2 + z^2 = (fx)^2;$$

and, by revolution round Oy , the surface

$$y = f\{\sqrt{(x^2 + z^2)}\};$$

while the curve given by the implicit relation $F(x, y) = 0$ will sweep out the surfaces

$$F(x, \sqrt{y^2 + z^2}) = 0, \text{ and } F(\sqrt{x^2 + z^2}, y) = 0.$$

We may also write the relation (A) in the form

$$F(x, y, z) = 0 \dots \dots \dots (B),$$

when an *implicit* relation connects x , y , z (§ 13).

Thus $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$

is the equation of an *ellipsoid* (fig. 43); but

$$(x/a)^2 + (y/b)^2 = 1$$

is now in space the equation of a *cylinder* standing on an elliptic base, the axis being parallel to Oz .

And generally, in space, the equations $y = fx$ or $F(x, y) = 0$ will represent cylindrical surfaces, the cross section being the graphs of the corresponding plane curve.

124. Notation of Partial Differential Coefficients.

Now returning to equation (A), where, to fix the ideas, Oz is supposed vertical, and making a section of the surface by a vertical plane parallel to the axis of x (fig. 43); then the tangent of the slope to the horizon of the curve of section will be the d.c. of z with respect to x , keeping y constant; this tangent of slope is expressed by

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f(x, y)}{\partial x} \left(\text{generally abbreviated to } \frac{\partial f}{\partial x} \right),$$

and this is called the *partial* d.c. of z with respect to x , ∂ being used when the differentiation is *partial*.

Again, if a section of the surface is made by a vertical plane parallel to the axis of y (fig. 43), then the tangent of the slope of the curve of section will be the d.c. of z with respect to y , keeping x constant; and is expressed by

$$\frac{\partial z}{\partial y} \text{ or } \frac{\partial f(x, y)}{\partial y} \left(\text{abbreviated to } \frac{\partial f}{\partial y} \right).$$

Now, to find dz/dt , the rate of increase of z , when x and y increase at given rates dx/dt and dy/dt , let Δx and Δy denote small finite increments of x and y , and Δz the corresponding increment of z . Then

$$z + \Delta z = f(x + \Delta x, y + \Delta y), \text{ where } z = f(x, y),$$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y),$$

$$\frac{\Delta z}{\Delta t} = \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y)}{\Delta t}$$

$$= \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x} \frac{\Delta x}{\Delta t}$$

$$+ \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \frac{\Delta y}{\Delta t}.$$

Proceeding to the limit, when Δt , Δx , and Δy are made indefinitely small, then

$$\lim \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{\partial f(x, y)}{\partial y} = \frac{\partial z}{\partial y};$$

and $\lim \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x}$

$$= \lim \frac{\partial f(x, y + \Delta y)}{\partial x} = \frac{\partial f(x, y)}{\partial x} = \frac{\partial z}{\partial x};$$

so that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \dots \dots \dots (1),$$

or $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

in the notation of Differentials (§ 11).

Thus we deduce from the relation

$$z = (\sqrt{X} - \sqrt{Y})/(x - y),$$

where $X = ax^2 + 2bx + c$, $Y = ay^2 + 2by + c$,

$$\frac{2dz}{a - z^2} = \frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}};$$

and more generally, from the results of Ex. 39, p. 82,

$$\frac{ds}{\sqrt{(4s^3 - g_2s - g_3)}} = \frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}},$$

where $X = ax^4 + 4bx^3 + 6cx^2 + 4b'x + a'$, ...,

when $s = \frac{1}{4} \left(\frac{\sqrt{X} - \sqrt{Y}}{x - y} \right)^2 - \frac{1}{4}a(x + y)^2 - b(x + y) - c$.

Differentiating equation (1) again, according to this rule,

$$\begin{aligned} \frac{d^2z}{dt^2} &= \frac{\partial z}{\partial x} \frac{d^2x}{dt^2} + \frac{\partial z}{\partial y} \frac{d^2y}{dt^2} \\ &+ \frac{\partial^2 z}{\partial x^2} \frac{dx^2}{dt^2} + 2 \frac{\partial^2 z}{\partial x \partial y} \frac{dx}{dt} \frac{dy}{dt} + \frac{\partial^2 z}{\partial y^2} \frac{dy^2}{dt^2} \dots \dots \dots (2); \end{aligned}$$

and so on, for any number of differentiations.

Similarly from the implicit relation (B) connecting x , y , z , we can deduce by successive differentiation the first, second, ... derived equations, in the form

$$\frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0, \dots$$

125. *Derived Equations of Implicit Relations.*

Suppose $z = c$, a constant; then $f(x, y) = c$ is the implicit relation (§ 13) connecting x and y along the curve of section (a contour line) of the surface $z = f(x, y)$ made by the (horizontal) plane $z = c$, and is therefore the equation of the curve of section.

Then, along this curve, $dz/dt = 0$, and therefore

$$\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 0;$$

the *first derived equation* (§ 13) with t for variable.

This can be established independently; for if

$$f(x, y) = c, \text{ and } f(x + \Delta x, y + \Delta y) = c,$$

then $f(x + \Delta x, y + \Delta y) - f(x, y) = 0$;

which can be written

$$\frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x} + \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \frac{\Delta y}{\Delta x} = 0;$$

reducing, when Δx and Δy are indefinitely small, to

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0.$$

The *second derived equation* is obtained by differentiating again with respect to x , and is therefore

$$\frac{\partial f}{\partial y} \frac{d^2 y}{dx^2} + \frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial^2 f}{\partial x \partial y} \frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2} \frac{dy^2}{dx^2} = 0;$$

and so on, for the third, fourth, ... derived functions.

We see now that the equations of the tangent and normal of a curve given by the implicit relation

$$f(x, y) = c, \text{ or } f(x, y, c) = 0,$$

can be written

$$(x' - x) \frac{\partial f}{\partial x} + (y' - y) \frac{\partial f}{\partial y} = 0, \quad (x' - x) \Big/ \frac{\partial f}{\partial x} = (y' - y) \Big/ \frac{\partial f}{\partial y},$$

while the radius of curvature

$$\rho = \left\{ \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right\}^{\frac{3}{2}} \Big/ \left\{ \frac{\partial^2 f}{\partial x^2} \left(\frac{\partial f}{\partial y} \right)^2 - 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y^2} \left(\frac{\partial f}{\partial x} \right)^2 \right\}.$$

Different values of $z = c$ will give the different contour lines of the ground on a plan, and will give an idea of the shape of the ground.

We may also suppose equation (A) to represent the outside surface of a ship, and that $f(x, y) = c$ represents, for different values of c , the water line curves of the ship, floating upright at a varying draught of water c ; or that it represents cross sections of the ship perpendicular to the keel at a distance c from the midship section.

126. *Expansion of a Function of two or more Independent Variables.*

Let h and k now represent the small finite increments of x and y in the function $z=f(x, y)$; and let z_1 denote the new value of z .

Then $z_1=f(x+h, y+k)$; which, expanded by Taylor's Theorem (§ 106), and first in powers of h , gives

$$z_1=f(x, y+k)+h\frac{\partial}{\partial x}f(x, y+k)+\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}f(x, y+k)+\dots;$$

and then, with each term expanded in powers of k , and written diagonally,

$$\begin{aligned} z_1=f(x, y)+h\frac{\partial f}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}+\dots \\ +k\frac{\partial f}{\partial y}+hk\frac{\partial^2 f}{\partial x\partial y}+\dots \\ +\frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}+\dots \\ +\dots \end{aligned}$$

the general term being

$$\frac{1}{n!}\left(h^n\frac{\partial^n f}{\partial x^n}+nh^{n-1}k\frac{\partial^n f}{\partial x^{n-1}\partial y}+\dots+k^n\frac{\partial^n f}{\partial y^n}\right),$$

which may be written in the symbolical form (§ 106)

$$\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x, y);$$

and the remainder R may be written (§ 114)

$$\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x+\theta h, y+\theta k).$$

Since it is immaterial whether we expand, first with respect to x and then with respect to y , or in reverse order; or, geometrically, whether we take the step h parallel to Ox first, and then the step k parallel to Oy , or *vice versa*; it follows that the order of partial differentiation is immaterial, or that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}, \text{ and generally } \frac{\partial^{p+q} z}{\partial x^p \partial y^q} = \frac{\partial^{p+q} z}{\partial y^q \partial x^p}.$$

This may be proved independently from the definition;

$$\text{for } \frac{\partial z}{\partial y} = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k},$$

$$\frac{\partial^2 z}{\partial x \partial y} = \lim_{h, k \rightarrow 0} \frac{f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y)}{hk},$$

and $\frac{\partial^2 z}{\partial y \partial x}$ is the limit of the same expression.

The general form of the expansion can be more easily perceived by putting $Ft = f(x+ht, y+kt)$, and expanding Ft in powers of t by Maclaurin's Theorem; afterwards putting $t=1$.

$$\text{For } Ft = h \frac{\partial Ft}{\partial x} + k \frac{\partial Ft}{\partial y},$$

$$F''t = h^2 \frac{\partial^2 Ft}{\partial x^2} + 2hk \frac{\partial^2 Ft}{\partial x \partial y} + k^2 \frac{\partial^2 Ft}{\partial y^2} = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 Ft,$$

when written symbolically (§ 106); and generally

$$F^{(n)}t = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n Ft;$$

so that, putting $t=1$ in Maclaurin's Theorem,

$$\begin{aligned} f(x+h, y+k) &= f(x, y) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(x, y) + \dots \\ &\quad + \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x, y) \\ &\quad + \frac{1}{(n+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(x+\theta h, y+\theta k). \end{aligned}$$

Expressed in a general symbolical form (§ 106),

$$f(x+h, y+k) = \exp \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(x, y),$$

a theorem capable of immediate generalization to the expansion of a function of any number of variables.

As an illustration we may expand $(x+h)^m(y+k)^n(z+l)^p \dots$ in ascending powers of h, k, l, \dots

127. *Maximum and Minimum Values of a Function of Two Independent Variables.*

To determine the maximums and minimums of $f(x, y)$, a function of two independent variables x and y , suppose the surface, which is its graph, to be cut into a series of contour lines by the parallel (horizontal) planes $z = c$.

Then, at the summit of a hill, where z has a maximum value, the contour line shrinks into a point, and

$$\frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial y} = 0.$$

The contour line a little lower down in a small closed curve, and the corresponding horizontal plane cuts off a small *cap* from the surface.

At the bottom of a lake, where z has a minimum value, the contour line again shrinks into point, and

$$\frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial y} = 0;$$

and the contour line a little above is a small closed curve; the corresponding horizontal plane cutting off a small *cup* from the surface.

Starting from the top of a hill and going down, as shown when a flood of water is running off, the contour lines enlarge, till a *pass* is reached, where two hills meet; here the contour lines cross, and

$$\frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial y} = 0;$$

but z is not an absolute maximum or minimum; for z is a minimum with respect to the two adjacent hills, but a maximum with respect to the two adjacent valleys.

Again, starting from the bottom of a lake and going up, the contour lines enlarge, as when a flood is rising, till a *bar* is reached, where two depressed regions or lake districts meet; here the contour lines cross, and again

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

but z is neither a maximum or minimum.

These geometrical considerations are useful in the problem of finding the maximum or minimum of a function of two independent variables.

(Cayley, *Contour and Slope Lines*, *Phil. Mag.*, 1859;

Maxwell, *Hills and Dales*, *Phil. Mag.*, 1870;

Rev. E. Hill, *Messenger of Mathematics*, vol. v.).

*128. *The Indicatrix. Dupin's Theorems for Normal Sections of a Surface.*

By changing the origin O to a point on the surface (A) where $\partial z/\partial x = 0$ and $\partial z/\partial y = 0$, we may write the equation of the surface in the form (§ 126)

$$z = \frac{1}{2}(ax^2 + 2hxy + by^2) + \text{higher powers, etc., of } x \text{ and } y;$$

where a, h, b , are the values of $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ at O .

This is now the general form of the equation of a surface in the neighbourhood of a point, when the tangent plane is taken as the plane xOy , and the normal Oz as the axis of z ; analogous to the method of § 96, where the tangent and normal at any point of a plane curve are taken as coordinate axes.

Neglecting terms in x and y of a degree higher than the second, then the equation

$$z = \frac{1}{2}(ax^2 + 2hxy + by^2)$$

represents a *paraboloid*, called the *osculating paraboloid*, or *paraboloid of curvature* at the point.

Sections of this paraboloid by parallel planes $z=c$ close to the tangent plane, which are very approximately sections of the surface, will be similar conic sections

$$ax^2 + 2hxy + by^2 = 2c;$$

and in the tangent plane the similar conic section

$$ax^2 + 2hxy + by^2 = 1$$

is called the *Indicatrix* (Dupin).

If $ab - h^2$ is positive, the indicatrix is an ellipse, and a plane $z=c$, close to the tangent plane, will, if it meets the surface, cut off a small cup from the surface; the point on the surface is then called a *synclastic* or *cup* point; as at the top of a hill, or bottom of a valley; so that the contour lines in the neighbourhood are approximately similar ellipses; and the osculating paraboloid is *elliptic*.

But if $ab - h^2$ is negative, the indicatrix is a hyperbola, and the tangent plane cuts the surface in two lines crossing at the point of contact; such a point is called an *anticlastic* or *saddle* point, such as a pass or bar; so that the contour lines in the neighbourhood are approximately similar hyperbolas; and the osculating paraboloid is *hyperbolic*.

Making a normal section of the surface and its osculating paraboloid by a plane in which $x = r \cos \theta$, $y = r \sin \theta$, then $\text{lt } 2z/r^2 = a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta$.

But $\text{lt } 2z/r^2$ is the curvature $1/\rho$ of the normal section of the surface (§ 96), so that ρ the radius of curvature of a normal section is equal to the square of the corresponding radius vector of the indicatrix, since

$$1/\rho = a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta.$$

If ρ' is the radius of curvature of the normal section at right angles to the first, then

$$1/\rho' = a \sin^2 \theta - 2h \sin \theta \cos \theta + b \cos^2 \theta,$$

so that $1/\rho + 1/\rho' = a + b$, a constant ;

a theorem due to Euler ; and if $\theta = 0$, $1/\rho = a$, $1/\rho' = b$.

We may revolve the axes Ox and Oy round Oz so as to make h disappear ; and now the equation is of the form

$$z = \frac{1}{2}x^2/R + \frac{1}{2}y^2/R' + \dots;$$

and R, R' the radii of curvature of the normal sections xOz, yOz , are called the *principal* radii of curvature of the surface, and Ox and Oy are called the directions of the *lines of curvature* at O ; and now

$$1/\rho = \cos^2 \theta / R + \sin^2 \theta / R'$$

for a normal section making an angle θ with Ox .

The quantity $a + b$, or $1/\rho + 1/\rho'$, or $1/R + 1/R'$ is called the *curvature of the surface* (Sir W. Thomson, *Capillary Attraction*) ; and to measure this curvature a *spherometer* is employed, consisting of a small plate or table, resting on the surface by three feet at the corners of an equilateral triangle, with a fourth foot at the centre which can be screwed down to touch the surface by means of a graduated micrometer screw.

Now if c denotes the radius of the spherometer, and z_1, z_2, z_3 the distances of the three feet from the tangent plane at the central foot, then for any angle of orientation θ ,

$$2z_1/c^2 = a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta$$

$$2z_2/c^2 = a \cos^2(\theta + \frac{2}{3}\pi) + \dots\dots$$

$$2z_3/c^2 = a \cos^2(\theta + \frac{4}{3}\pi) + \dots\dots$$

so that the distance of the central foot from the plane of the three feet

$$= \frac{1}{3}(z_1 + z_2 + z_3) = \frac{1}{4}c^2(a + b) ;$$

which is independent of the orientation θ , and the same as for a sphere of equal curvature.

*129. *Meunier's Theorem for oblique sections of a Surface.*

When we draw a normal plane to a sphere at any point, cutting the sphere in a great circle of radius R , the radius of the sphere, and when we draw through the same tangent line an oblique plane, inclined at an angle θ to the normal plane, cutting the sphere in a small circle, then the radius of this small circle is obviously $R \cos \theta$.

A similar theorem, called Meunier's Theorem, connects the radius of curvature of any oblique section of a surface with the normal section having the same tangent line; for if the oblique plane $z = x \tan \theta$ cuts the surface

$$z = \frac{1}{2}(ax^2 + 2hxy + by^2) + \dots,$$

then the curvature of the section of the surface is $\frac{1}{2} \sec \theta / x^2$, with $y = 0$, which is equal to $\frac{1}{2} \sec \theta$; so that the radius of curvature is $\cos \theta / \alpha$.

*130. *Solid Geometry.*

Further geometrical applications would lead us beyond the scope of this book; we may however enunciate a number of useful definitions and propositions in Solid Geometry, for the student either to establish himself, or to refer for their demonstration to the treatises of Smith, Frost, or Salmon.

A *line of curvature* on a surface is one whose tangent is an axis of the indicatrix at every point.

Two lines of curvature pass at right angles through an ordinary point on a surface.

A point where the principal radii of curvature R and R' are equal is called an *umbilicus*.

Lines of curvature converge to an umbilicus.

In the neighbourhood of a point on a surface the normals along a line of curvature ultimately intersect,

but other normals pass through two (focal) lines at right angles to one another, one at each centre of principal curvature. These focal lines are seen experimentally when a narrow beam of light is received directly on a screen at a variable distance; rays of light being always capable of being cut orthogonally by a surface.

When normals are drawn round a small closed contour of surface A , described on a surface round a point, and parallel normals are drawn on a sphere of radius c , then the corresponding contour on the sphere has an area $c^2 A/RR'$; this is easily established if the contour is bounded by the neighbouring lines of curvature.

The quantity $1/RR'$ is called Gauss's *measure of curvature*; if the surface is bent in any manner without stretching, then $1/RR'$ is unaltered.

Tortuous Curves. Two surfaces intersect in a line, called a *tortuous curve* if it does not lie in a plane.

The two tangent planes of the surfaces intersect in the *tangent line* of the curve, while the two normal lines of the surfaces lie in the *normal plane* of the curve.

The properties of a tortuous curve are investigated, as in § 90, by considering the curve as the limit of a twisted polygon of short links or chords.

The plane through two consecutive links is called the *osculating plane*; the normal line in this plane is called the *principal normal*, and the normal line perpendicular to this plane is called the *binormal*.

The centre of curvature in the osculating plane is called the *centre of principal curvature*; and $\rho = ds/d\psi$ is called the *radius of principal curvature*, $\Delta\psi$ denoting the angle between consecutive normal planes.

Accents now denoting differentiation with respect to the arc s , we shall find, as in § 94,

$$1/\rho^2 = x''^2 + y''^2 + z''^2;$$

$$\alpha = x + \rho^2 x'', \beta = y + \rho^2 y'', \gamma = z + \rho^2 z'',$$

α, β, γ denoting the coordinates of the centre of principal curvature; and this point will be the foot of the perpendicular drawn from the point P on the line AB , joining A and B the centres of curvature of the normal sections of any two surfaces intersecting in the curve.

The normal planes of the curve carve out a surface called the *polar developable*.

Three consecutive normal planes intersect in a point which is called the *centre of spherical curvature*.

For if normal planes are drawn through P_1, P_2, P_3 , the middle points of three consecutive links of a tortuous chain (fig. 35), then their point of intersection Q_2 will be equidistant from the four ends of the three links, and therefore Q_2 will be the centre of a sphere passing through these points, and R , the radius of this sphere, is called the *radius of spherical curvature*.

The locus of these points Q is called the *edge of regression* of the polar developable.

The angle between two consecutive osculating planes being denoted by $\Delta\tau$, then $d\tau/ds$ is called the *torsion*; and denoting it by $1/\sigma$, then σ is called the *radius of torsion*; we shall also find

$$R^2 = \rho^2 + (d\rho/d\tau)^2 = \rho^2 + \sigma^2(d\rho/ds)^2.$$

An *osculating helix* can be drawn having the same ρ and σ as the curve at a point; the axis of the helix will lie along the shortest distance between consecutive principal normals; and the radius and pitch of the helix will be

$$\rho\sigma^2/(\rho^2 + \sigma^2) \text{ and } \rho^2\sigma/(\rho^2 + \sigma^2).$$

A *geodesic* on a surface is the curve assumed by a stretched thread between two points, and therefore occupying the shortest distance; the osculating plane of a geodesic is normal to the surface.

If tangent planes of a curve are drawn perpendicular to the principal normal, these planes will carve out a surface called the *rectifying developable*, and the curve will be a *geodesic* on this surface.

*131. *Functional and Differential Equations of a Surface.*

We have already written down in § 123 the general equation of a surface of revolution, when the axis coincides with one of the coordinate axes; and now when the axis of the surface passes through the origin and makes angles α, β, γ with the coordinate axes Ox, Oy, Oz , the general equation of the surface is of the form

$$x^2 + y^2 + z^2 = f(x \cos \alpha + y \cos \beta + z \cos \gamma) \dots \dots (1),$$

since for any point P on the surface, OP and therefore OP^2 must be some function of the distance of P from the plane through O perpendicular to the axis of revolution; and this perpendicular distance is equal to

$$x \cos \alpha + y \cos \beta + z \cos \gamma,$$

as is seen by projecting OP in the axis; and now (1) is called the *functional equation* of a surface of revolution.

By partial differentiation with respect to x and y ,

$$2x + 2z \frac{\partial z}{\partial x} = f'(x \cos \alpha + y \cos \beta + z \cos \gamma) \left(\cos \alpha + \frac{\partial z}{\partial x} \cos \gamma \right);$$

$$2y + 2z \frac{\partial z}{\partial y} = f'(x \cos \alpha + y \cos \beta + z \cos \gamma) \left(\cos \beta + \frac{\partial z}{\partial y} \cos \gamma \right);$$

and then, eliminating $f'(x \cos \alpha + \dots)$ by division, we find

$$(y \cos \gamma - z \cos \beta) \frac{\partial z}{\partial x} + (z \cos \alpha - x \cos \gamma) \frac{\partial z}{\partial y} = x \cos \beta - y \cos \alpha,$$

a *partial differential equation*, since it involves partial derivatives, called the differential equation of surfaces of revolution.

Conversely, the solution of this partial differential equation is the functional equation (1).

The functional equation of a cylinder, having the generating lines in the same direction as the axis of the surface of revolution, may be written, in the implicit form,

$$F(y \cos \gamma - z \cos \beta, z \cos \alpha - x \cos \gamma) = 0 \dots \dots \dots (2),$$

from which we derive as before

$$\cos \alpha \frac{\partial z}{\partial x} + \cos \beta \frac{\partial z}{\partial y} = \cos \gamma,$$

the differential equation of a *cylindrical surface*.

The general functional equation of a *cone*, with vertex at O , will be $F(z/x, y/x) = 0$, or $z/x = f(y/x)$; with the corresponding differential equation

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z.$$

These differential equations are of the form

$$P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} = R,$$

where P, Q, R are functions of x, y, z ; this is called the general *partial differential equation of the first order*, and it represents surfaces built up of the curves given by

$$dx/P = dy/Q = dz/R;$$

and if $u = a, v = b$ be any two surfaces intersecting in one of these curves, the general solution of this partial differential equation is a functional equation, $F(u, v) = 0$.

In partial differential equations it is customary to write p and q for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, and r, s, t for $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$; but we must be careful to distinguish this use of r from its use as denoting the radius vector in polar coordinates.

Thus it is proved in Treatises on Solid Geometry that the principal radii of curvature R and R' of a surface (§ 128) are the roots of the quadratic equation

$$(rt - s^2)R^2 + \{(1 + p^2)t - 2pqs + (1 + q^2)r\}(1 + p^2 + q^2)R + (1 + p^2 + q^2)^2 = 0.$$

Hence the curvature of the surface (§ 128) is zero if

$$(1 + p^2)t - 2pqs + (1 + q^2)r = 0,$$

as we see verified experimentally in a soap-bubble film (C. V. Boys, *Soap Bubbles*); for instance in the surfaces

- (i.) $z = a \tan^{-1} y/x$, (ii.) $z/a = \cosh^{-1} \{ \sqrt{(x^2 + y^2)}/a \}$,
 (iii.) $e^z = \cos x / \cos y$, (iv.) $\sin mz = \sinh mx \sinh my$.

Gauss's measure of curvature

$$1/RR' = (rt - s^2)/(1 + p^2 + q^2)^2;$$

and this is zero, if $rt - s^2 = 0$; a partial differential equation derivable from $F(p, q) = 0$, and representing a *developable* surface, that is a surface which can be flattened out into a plane, without stretching.

It is readily proved that if a flexible inextensible surface in the form of the film in fig. 16 is cut open along a meridian curve AP , and the circle AA' is pulled out straight, the surface will assume the form of a uniform right screw surface, given by the equation $z = a \tan^{-1} y/x$, instead of its original form $z = a \cosh^{-1} \{ \sqrt{(x^2 + y^2)}/a \}$; and that Gauss's measure of curvature in the two surfaces is the same at corresponding points.

Similarly, the modified surface

$$z = b \cosh^{-1} \{ \sqrt{(x^2 + y^2)/a} \}$$

will, when the circle AA' is pulled out straight, assume the form of a uniform skew screw surface, swept out by generating lines making an angle $\sin^{-1}b/a$ with the axis of the screw, as in a V shaped thread; while if the circle AA' is bent into a helix of pitch b on a cylinder of radius $\sqrt{(a^2 - b^2)}$, the generating lines become perpendicular to the axis of the cylinder.

But if the radius of the circle AA' is changed from a to b , the surface will become the *hyperboloid*

$$\frac{z^2}{b^2} = \frac{x^2 + y^2}{a^2 - b^2} - 1.$$

Generally, one inextensible surface of revolution can be wrapped upon another surface of revolution, when the meridian curve AP of the first surface is a plane section of a cylinder, and the meridian curve AP' of the second is the curve which AP becomes when the cylinder is flattened into a plane.

For if the equal arcs AP , AP' are denoted by s , and if y , y' denote the corresponding ordinates of P , P' with respect to Ox , the line of intersection of the plane of AP with a plane base or cross section of the cylinder, with which it makes an angle α , then $y' = y \sin \alpha$; and the first surface can be applied to the second, so that the meridian arc AP becomes AP' without change of length, but the circular cross section of the surface at P becomes contracted into a circle, smaller in the ratio of $\sin \alpha$ to 1.

In the case above, the base of the cylinder is a common catenary, and the curve AP is a section of the cylinder made by a plane inclined at an angle $\cos^{-1}b/a$ to the base, and is therefore a modified catenary; and when the

cylinder is developed into a plane, it is readily seen that the curve AP becomes changed into a hyperbola AP' .

As another illustration, we may take the base of the cylinder a circle; and now the curve AP is an ellipse of excentricity $e = \sin \alpha$, which by revolution round Ox , the minor axis in which it cuts a circular cross section of the cylinder, sweeps out an oblate spheroid; and this oblate spheroid is applicable on a surface of which the meridian curve AP' is a *sinusoid* (§ 20). We may therefore cut away $1 - e$ of the surface of the prolate spheroid by two meridian cuts, and join the edges together, when the new surface will be formed by the revolution of a sinusoid.

Examples on Partial Differentiation.

(1) Deduce the differential relation

$$\frac{dz}{\sqrt{(1 - cz^2 + z^4)}} = \frac{dx}{\sqrt{(1 - cx^2 + x^4)}} + \frac{dy}{\sqrt{(1 - cy^2 + y^4)}}$$

from the integral relation

$$z = \{x\sqrt{(1 - cy^2 + y^4)} + y\sqrt{(1 - cx^2 + x^4)}\} / (1 - x^2y^2).$$

*(2) Denoting $Ax^3 + 3Bx^2 + 3Cx + D$ by X, \dots , prove that

$$(y - z)X^{\frac{1}{3}} + (z - x)Y^{\frac{1}{3}} + (x - y)Z^{\frac{1}{3}} = 0$$

leads to the relations, integral and differential,

$$\{Axyz + B(yz + zx + xy) + C(x + y + z) + D\}^3 = XYZ,$$

$$\text{and} \quad X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy + Z^{-\frac{2}{3}}dz = 0.$$

(MacMahon, *Quart. Journal of Math.* XIX. and XX.)

*(3) Deduce the differential relation

$$\frac{d\mu}{\sqrt{(1 - \kappa^2 \sin^2 \mu)}} = \frac{d\theta}{\sqrt{(1 - \kappa^2 \sin^2 \theta)}} + \frac{d\phi}{\sqrt{(1 - \kappa^2 \sin^2 \phi)}}$$

from $\cos \mu = \cos \theta \cos \phi - \sin \theta \sin \phi \sqrt{(1 - \kappa^2 \sin^2 \mu)}$.

Discuss the degenerate cases of $\kappa = 0$, and $\kappa = 1$.

(4) Determine the max. and min. of x, y , or z in

$$x^2 + y^2 + z^2 + yz + zx + xy = 2a^2.$$

(5) Prove that the partial differential equation

(i.) $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ is satisfied by $u = Ct^{-\frac{1}{2}} \exp(-x^2/4kt)$;

(ii.) $\frac{\partial u}{\partial t} = k \left(a^2 \frac{\partial^2 u}{\partial x^2} + b^2 \frac{\partial^2 u}{\partial y^2} \right)$

by $u = Ct^{-1} \exp\left\{-\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)/4kt\right\}$;

(iii.) $\frac{\partial u}{\partial t} = k \left(a^2 \frac{\partial^2 u}{\partial x^2} + b^2 \frac{\partial^2 u}{\partial y^2} + c^2 \frac{\partial^2 u}{\partial z^2} \right)$

by $u = Ct^{-\frac{3}{2}} \exp\left\{-\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)/4kt\right\}$.

(6) * Given $V = (1 - 2\mu h + h^2)^{-\frac{1}{2}} = 1 + \Sigma h^n P_n$,

prove that

$$\frac{\partial}{\partial h} \{(1 - \mu h) V\} = -(1 - \mu^2) \frac{\partial V}{\partial \mu}, \quad \frac{\partial}{\partial \mu} \{(1 - \mu h) V\} = h^2 \frac{\partial V}{\partial h};$$

and thence $h^2 \frac{\partial^2 V}{\partial h^2} + 2h \frac{\partial V}{\partial h} + \frac{\partial}{\partial \mu} \{(1 - \mu^2) \frac{\partial V}{\partial \mu}\} = 0$,

and $\frac{d}{d\mu} \left\{ (1 - \mu^2) \frac{dP_n}{d\mu} \right\} + n(n+1)P_n = 0$.

(P_n is called the *zonal surface harmonic* of the n^{th} degree, and $h^n P_n$ the *zonal solid harmonic*.)

Prove also that

$$\frac{1}{V^2} \frac{\partial V}{\partial h} = (\mu - h) V, \quad \frac{1}{V^2} \frac{\partial V}{\partial \mu} = h V;$$

and deduce

(i.) $(n+1)P_{n+1} - (2n+1)\mu P_n + nP_{n-1} = 0$;

(ii.) $P'_{n+1} - 2\mu P'_n + P'_{n-1} = P_n$,

the accent denoting differentiation with respect to μ ;

(iii.) $\mu P'_n - P'_{n-1} = nP_n$;

(iv.) $P'_{n+1} - P'_{n-1} = (2n+1)P_n$;

(v.) $(\mu^2 - 1)P'_n = n(\mu P_n - P_{n-1}) = (n+1)(P_{n+1} - \mu P_n)$
 $= n(n+1)(P_{n+1} - P_{n-1})/(2n+1).$

- * (7) Putting $\mu h = z$, $h^2 = r^2 + z^2$, and denoting $h^n P_n$ by Z_n , prove that

$$\frac{\partial Z_n}{\partial z} = n Z_{n-1}, \quad \frac{\partial^p Z_n}{\partial z^p} = n(n-1)\dots(n-p+1)Z_{n-p};$$

and thence that writing $z+c$ for z changes Z_n into

$$Z_n + ncZ_{n-1} + \frac{n(n-1)}{1 \cdot 2} c^2 Z_{n-2} + \dots + nc^{n-1} Z_1 + c^n.$$

- (8) Prove that the differential equation

$$\frac{\partial^2 u}{\partial x^2} - q^2 u = \frac{h^2}{x^2} \frac{\partial^2 u}{\partial h^2}$$

is satisfied by $u = \exp q \sqrt{(x^2 + xh)}$; and

$$\frac{\partial^2 u}{\partial x^2} \pm q^2 u = \frac{h^2}{x^2} \frac{\partial^2 u}{\partial x^2} \text{ by } A_{\cosh}^{\cos} q \sqrt{(x^2 + xh)} + B_{\sinh}^{\sin} q \sqrt{(x^2 + qh)}$$

(Glaisher, *Phil. Trans.*, 172); and thence deduce the solution of (§ 88)

$$\frac{1}{u} \frac{d^2 u}{dx^2} = \frac{n(n+1)}{x^2} \pm q^2.$$

- (9) The differential equation which results from the elimination of the arbitrary functions from

(i.) $z = f(x+ay) + F(x-ay)$ is $t - a^2 s = 0$;

(ii.) $z = \phi x \psi y$ is $sz = pq$;

(iii.) $\phi x + \psi y + \chi z = 0$ is $\frac{\partial}{\partial y} \left(\frac{r}{p} - \frac{s}{q} \right) = 0$, or $\frac{\partial}{\partial x} \left(\frac{s}{p} - \frac{t}{q} \right) = 0$.

- (10) Prove that the shortest distance between two consecutive curves of the system $F(x, y, c) = 0$ is

$$\Delta c \frac{\partial F}{\partial c} / \sqrt{\left\{ \left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 \right\}}.$$

Prove that if $f(x, y) = c$ is the equation of a system of *parallel* curves (§ 95), then

$$\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 = 1.$$

*132. *Change of Variables.*

Sometimes we require to change to new independent variables, the polar coordinates r and θ (§ 22), instead of x and y ; and now the system of coordinates r, θ, z are called *cylindrical* or *columnar* coordinates, and are suitable to employ in problems in which symmetry about an axis, taken as Oz , exists.

With $x = r \cos \theta$, $y = r \sin \theta$, differentiate with respect to the *new* variables, and then solve; this will be found an infallible method, when the old variables are given explicitly in terms of the new variables; then

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = p \cos \theta + q \sin \theta, \\ \frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = -pr \sin \theta + qr \cos \theta;\end{aligned}$$

so that, by solution of these equations,

$$p = \frac{\partial z}{\partial r} \cos \theta - \frac{\partial z}{r \partial \theta} \sin \theta, \quad q = \frac{\partial z}{\partial r} \sin \theta + \frac{\partial z}{r \partial \theta} \cos \theta.$$

We should proceed in the same manner if the old variables x and y were given explicitly as functions of new variables u and v by any relations of the form

$$x = f(u, v), \quad y = F(u, v).$$

Then, differentiating with respect to the new variables u and v ,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial f}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial F}{\partial u} \dots \dots \dots (1),$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial f}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial F}{\partial v} \dots \dots \dots (2).$$

It is convenient to employ the notation

$$\frac{\partial(x, y)}{\partial(u, v)} \text{ for } \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u};$$

and this function is called the *Jacobian* of x, y with respect to u, v ; and now, by solution of the preceding equations, (1) and (2), we find

$$\frac{\partial z}{\partial x} = \frac{\partial(z, F)}{\partial(u, v)} / \frac{\partial(f, F)}{\partial(u, v)}, \quad \frac{\partial z}{\partial y} = \frac{\partial(f, z)}{\partial(u, v)} / \frac{\partial(f, F)}{\partial(u, v)}.$$

Further differentiation of equations (1) and (2) with respect to u and v will give equations for the determination of $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \dots$

But when the new variables are given explicitly in terms of the old variables; for instance, if we had taken

$$r = \sqrt{(x^2 + y^2)}, \quad \theta = \tan^{-1} y/x;$$

$$\text{then } \frac{\partial r}{\partial x} = \frac{x}{\sqrt{(x^2 + y^2)}} = \cos \theta, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{(x^2 + y^2)}} = \sin \theta,$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{\sin \theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{\cos \theta}{r};$$

and we should have found immediately, by differentiation with respect to the *old* variables,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r} \cos \theta - \frac{\partial z}{r \partial \theta} \sin \theta,$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial y} = \frac{\partial z}{\partial r} \sin \theta + \frac{\partial z}{r \partial \theta} \cos \theta.$$

Differentiating again, with respect to x and y , will determine $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \dots$; and we shall find that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{\partial z}{r \partial r} + \frac{\partial^2 z}{r^2 \partial \theta^2},$$

$$\frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 = \frac{\partial^2 z}{\partial r^2} \left(\frac{\partial z}{r \partial r} + \frac{\partial^2 z}{r^2 \partial \theta^2} \right) - \left(\frac{\partial^2 z}{r \partial r \partial \theta} - \frac{\partial z}{r^2 \partial \theta} \right)^2;$$

so that these expressions are unaltered by *orthogonal transformation*, that is, by a change to any other system of rectangular axes.

When the old and new variables are connected by implicit relations, it is immaterial whether we differentiate with respect to the old or the new system; but it is advisable to keep entirely to the system first chosen.

Now, with the implicit relations,

$$f(x, y, u, v) = 0, \quad F(x, y, u, v) = 0,$$

and differentiating for instance with respect to the old variables,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = 0,$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial x} = 0;$$

so that

$$\frac{\partial u}{\partial x} = \frac{\partial(f, F)}{\partial(v, x)} / \frac{\partial(f, F)}{\partial(u, v)},$$

$$\frac{\partial v}{\partial x} = \frac{\partial(f, F)}{\partial(x, u)} / \frac{\partial(f, F)}{\partial(u, v)},$$

and

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \\ &= \left\{ \frac{\partial z}{\partial u} \frac{\partial(f, F)}{\partial(v, x)} + \frac{\partial z}{\partial v} \frac{\partial(f, F)}{\partial(x, u)} \right\} / \frac{\partial(f, F)}{\partial(u, v)}. \end{aligned}$$

Similarly for $\frac{\partial z}{\partial y}$ and the higher partial derivatives.

It is sometimes necessary to permute the variables x, y, z , and to make, say x the dependent, and y, z the independent variables in a relation of the form (B) (§ 123),

$$F(x, y, z) = 0.$$

$$\text{Now} \quad \frac{\partial F}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial F}{\partial z} = 0;$$

while, with the former use of p, q, r, \dots ,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p = 0, \quad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q = 0;$$

so that $\frac{\partial x}{\partial z} = \frac{1}{p}, \quad \frac{\partial x}{\partial y} = -\frac{q}{p}.$

Similarly we find, with the notation of p. 273,

$$\frac{\partial^2 x}{\partial z^2} = -\frac{r}{p^3}, \quad \frac{\partial^2 x}{\partial y \partial z} = \frac{qr - ps}{p^3}, \quad \frac{\partial^2 x}{\partial y^2} = -\frac{q^2 r - 2pqs + p^2 t}{p^3};$$

Analogous to the Reciprocants of § 86, we obtain *Ternary Reciprocants*, functions of the partial derivatives p, q, r, s, t, \dots , the form of which is unaltered when the variables x, y, z are permuted in cyclical order.

(Elliott, *Proc. London Math. Society*, 1886-1889 ;
Forsyth, *Phil. Trans.*, 1889.)

Examples on Change of the Variables.

(1) Given $x = u + v, y = uv$, prove that

$$\frac{\partial z}{\partial x} = \left(u \frac{\partial z}{\partial u} - v \frac{\partial z}{\partial v} \right) / (u - v), \quad \frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \right) / (u - v).$$

(2) With $x = r \cosh u, y = r \sinh u$, prove that

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{\partial z}{r \partial r} - \frac{\partial^2 z}{r^2 \partial u^2}.$$

(3) Prove that, on changing to oblique coordinates ξ and η , the axes being inclined at an angle ω ,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial \xi^2} - 2 \cos \omega \frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2} \right) / \sin^2 \omega,$$

and write down the transformation of

$$\frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2.$$

(4) Prove that the operator $\exp \alpha \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$ changes

$f(x, y)$ into $f(x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha)$;
that is, turns the axes through an angle α .

(5) From $f(x, y, z) = 0, F(x, y, z) = 0$, prove that

$$\frac{dy}{dx} = \frac{\partial(f, F)}{\partial(z, x)} / \frac{\partial(f, F)}{\partial(y, z)}, \quad \frac{dz}{dx} = \frac{\partial(f, F)}{\partial(x, y)} / \frac{\partial(f, F)}{\partial(y, z)}.$$

(6) Prove that, if x, y and ξ, η are given functions of u, v ,

$$\frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{\partial(x, y)}{\partial(u, v)} \bigg/ \frac{\partial(\xi, \eta)}{\partial(u, v)},$$

where x, y on the left hand side are supposed expressed in terms of ξ, η .

(7) Given four functions p, v, θ, ϕ , of which two are independent, and E a function of v, ϕ , such that

$$dE = \theta dv - p d\phi;$$

prove that (Maxwell, *Theory of Heat*, p. 169)

$$(i.) \quad \frac{\partial v}{\partial \theta} (p \text{ constant}) = - \frac{\partial \phi}{\partial p} (\theta \text{ constant});$$

$$(ii.) \quad \frac{\partial v}{\partial \phi} (p \text{ constant}) = \frac{\partial \theta}{\partial p} (\phi \text{ constant});$$

$$(iii.) \quad \frac{\partial p}{\partial \theta} (v \text{ constant}) = \frac{\partial \phi}{\partial v} (\theta \text{ constant});$$

$$(iv.) \quad \frac{\partial p}{\partial \phi} (v \text{ constant}) = - \frac{\partial \theta}{\partial v} (\phi \text{ constant}).$$

*133. Conjugate Functions.

Two quantities u and v are called conjugate functions of x and y , when the complex quantities $u + iv$ and $x + iy$ are functions of each other, where i denotes $\sqrt{-1}$ (Maxwell, *Electricity*, vol. I., Chap. xii.).

Conjugate functions are useful in physical problems on the plane flow of liquid and electrical currents.

We denote $u + iv$ by w and $x + iy$ by z , for brevity; and now, if

$$w = fz, \text{ or } u + iv = f(x + iy),$$

$$\text{then} \quad \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = f'z,$$

$$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} = if'z;$$

$$\text{so that} \quad \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} = i \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x};$$

and equating the real and imaginary parts,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Consequently
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y};$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y};$$

so that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Denoting by J the Jacobian

$$\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \text{ or } \frac{\partial(u, v)}{\partial(x, y)},$$

$$J = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2, \text{ or } \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2;$$

and $J = f'(x+iy)f'(x-iy).$

Conversely, x and y are conjugate functions of u and v , given by

$$x + iy = f^{-1}(u + iv);$$

since $u + iv = f(x + iy)$, or $w = fz$;

and differentiating now with respect to u and v as independent variables,

$$1 = f'z \left(\frac{\partial x}{\partial u} + i \frac{\partial y}{\partial u} \right), \quad i = f'z \left(\frac{\partial x}{\partial v} + i \frac{\partial y}{\partial v} \right);$$

so that
$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \quad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u};$$

$$\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2} = 0, \quad \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} = 0;$$

and the Jacobian

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \\ &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2 = \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 \\ &= \frac{1}{f'(x+iy)f'(x-iy)} = \frac{1}{J}; \end{aligned}$$

or
$$\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1,$$

provided u and v are conjugate functions of x and y .

If ϕ and ψ denote conjugate functions of u and v , it stands to reason that ϕ and ψ are also conjugate functions of x and y ; but if ϕ is not a conjugate function, we shall

find
$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \frac{\partial(x, y)}{\partial(u, v)}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} \right) \frac{\partial(u, v)}{\partial(x, y)}.$$

Putting $u = \text{const.}$ or $v = \text{const.}$ gives a series of curves, the graphs of these functions with x and y as coordinates; and denoting by s_1 and s_2 the lengths of the arcs of these curves, we find

$$\frac{ds_2^2}{du^2} = \left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 = \frac{1}{J}, \quad \frac{ds_1^2}{dv^2} = \frac{1}{J},$$

so that
$$\frac{ds_2}{du} = \frac{ds_1}{dv} = J^{-\frac{1}{2}};$$

and the above conditions show that these curves u and v cut at right angles, thus forming, for equal increments of u and v , a network of orthogonal curves, the meshes of which are elementary squares; while it is readily proved that the curvature of these curves is $\partial J^{\frac{1}{2}}/\partial u$ and $\partial J^{\frac{1}{2}}/\partial v$.

Familiar instances are (i.) horizontal and vertical straight lines, (ii.) concentric circles and their radii, (iii.) confocal conics, (iv.) the dipolar circles of the stereographic projection of the meridians and parallels of the two hemispheres.

Examples on Conjugate Functions.

- (1) Prove that $u = r^n \cos n\theta$, $v = r^n \sin n\theta$ are conjugate functions of $x = r \cos \theta$, $y = r \sin \theta$; and that

$$\begin{aligned}\frac{\partial u}{\partial x} &= nr^{n-1} \cos(n-1)\theta, & \frac{\partial u}{\partial y} &= -nr^{n-1} \sin(n-1)\theta; \\ \frac{\partial^2 u}{\partial x^2} &= n(n-1) \cos(n-2)\theta = -\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial^2 u}{\partial x \partial y} &= -n(n-1)r^{n-2} \sin(n-2)\theta.\end{aligned}$$

(2) Prove that conjugate functions of x, y are given by

- (i.) $u = x^2 - y^2, \quad v = 2xy;$
- (ii.) $u = x^3 - 3xy^2, \quad v = 3x^2y - y^3;$
- (iii.) $u = \log \sqrt{(x^2 + y^2)}, \quad v = \tan^{-1} y/x;$
- (iv.) $u = \cosh x \cos y, \quad v = \sinh x \sin y;$
- (v.) $u = \frac{\sin x}{\cos x + \cosh y}, \quad v = \frac{\sinh y}{\cos x + \cosh y};$
- (vi.) $u = \tan^{-1}(\cos x / \sinh y), \quad v = \tanh^{-1}(\sin x / \cosh y);$
- (vii.) $u = \tan^{-1} \frac{2\alpha^n r^n \sin n\theta}{r^{2n} - \alpha^{2n}}, \quad v = \tanh^{-1} \frac{2\alpha^n r^n \cos n\theta}{r^{2n} + \alpha^{2n}}.$

(3) Determine the conjugate functions u, v of x, y from
 $w = z^2, \sqrt{z}, z^n, z^n - c^n, 1/z, (az + b)/(Az + B), \sin z,$
 $\tan z, \sec z, \sin^{-1} z, \tan^{-1} z, \sec^{-1} z, \log z, \exp z,$
 $\cosh z, \sinh z, \tanh z, \dots;$

and sketch the corresponding curves.

(4) Prove that $\log J$ is a conjugate function.

(5) Prove that

$$\begin{aligned}F\left(\frac{d}{dx}\right)y \sin(px + q) \\ = \sin(px + q)f\left(\frac{d}{dx}, p\right)y + \cos(px + q)\phi\left(\frac{d}{dx}, p\right)y,\end{aligned}$$

where, resolved into its conjugate functions

$$F(x + iy) = f(x, y) + i\phi(x, y).$$

(The statement at the foot of p. 142 is incorrect, and must be altered to this result.)

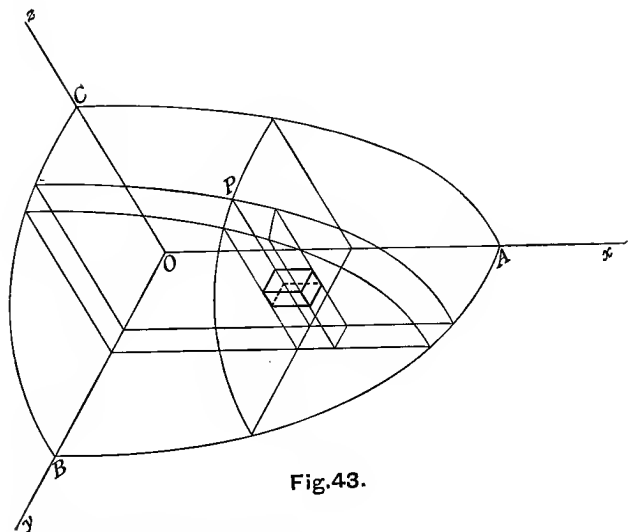


Fig. 43.

134. *Double Integration.*

Denote by V the volume of the solid which is bounded by the surface $z=f(x, y)$ and a cylinder standing on any base A in the plane of $z=0$, with generating lines parallel to the axis of z (fig. 43).

Then V , considered as a *fluent*, may be supposed generated by the motion of the *fluxion* $\partial V/\partial y$, an area moving with its plane perpendicular to the axis of y ; as seen for instance in gradually filling up the volume V with water, when the axis Oy is held vertical.

Again the area $\partial V/\partial y$, considered as a *fluent*, may be supposed generated by the motion of the ordinate z , moving parallel to the axis of x ; so that z is the *fluxion* of $\partial V/\partial y$ with respect to x , or

$$\frac{\partial^2 V}{\partial x \partial y} = z;$$

and integrating, doubly,

$$V = \iint z dx dy,$$

the integration extending over the area of the base A .

In the Infinitesimal method, the volume V may be supposed built up of filaments of length z and cross section $dx dy$.

Also if \bar{x} , \bar{y} , \bar{z} denote the coordinates of the centroid of the volume V ,

$$\bar{x}V = \iint xz dx dy, \quad \bar{y}V = \iint yz dx dy, \quad \bar{z}V = \iint \frac{1}{2}z^2 dx dy.$$

Applying this method to the determination of the volume of an octant of the ellipsoid (fig. 43)

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = 1,$$

and integrating, first with respect to x , the limits are 0 and $a\sqrt{(1-y^2/b^2)}$; and integrating afterwards with respect to y , the limits are 0 and b ; so that

$$V = \int_0^b \int_0^{a\sqrt{(1-y^2/b^2)}} c\sqrt{\{1-(x/a)^2-(y/b)^2\}} dx dy,$$

which, on substituting $\frac{x^2}{a^2} = \left(1 - \frac{y^2}{b^2}\right) \sin^2 \phi$, makes

$$\begin{aligned} V &= \int_0^b ac \left(1 - \frac{y^2}{b^2}\right) dy \int_0^{\frac{1}{2}\pi} \cos^2 \phi d\phi \\ &= \frac{1}{4}\pi ac \int_0^b \left(1 - \frac{y^2}{b^2}\right) dy = \frac{1}{6}\pi abc. \end{aligned}$$

Similarly we shall find

$$\bar{x}V = \frac{1}{16}\pi a^2 bc, \text{ so that } \bar{x} = \frac{3}{8}a; \text{ and } \bar{y} = \frac{3}{8}b, \bar{z} = \frac{3}{8}c,$$

by symmetry.

We may suppose z to denote the variable superficial density of a plane lamina, of area A ; and then the preceeding formulas will give V as the mass of the lamina and \bar{x} , \bar{y} the coordinates of its centre of mass.

If $z=c$, a constant, then V is the volume of a right cylinder on the base A , so that

$$V=c\iint dxdy, \text{ and } A=\iint dxdy;$$

and the area A may be considered as built up of the infinitesimal elements of area $dxdy$, the limits of integration being taken so as to include all the elements of area in A .

135. If the surface $z=f(x, y)$ becomes a plane, then

$$z=lx+my+n,$$

$$\text{and } V=\iint (lx+my+n)dxdy=(l\bar{x}+m\bar{y}+n)A=\bar{z}A,$$

where \bar{z} is the ordinate of the plane standing on the centroid of the base A ; this is the formula for the volume cut off a cylinder by a slant plane and a cross section, or by two slant planes.

By means of this principle we can calculate the volume of an earthwork dam across a valley, of which the contour lines are assumed to be parallel straight lines; also the volume of a groin formed by the intersection of two equal barrel vaults, crossing at right angles.

136. With polar (cylindrical) coordinates r , θ , and z , and double integration, the infinitesimal element of area enclosing a point (r, θ) in the plane of $z=0$ may be supposed bounded by circles of radii $r-\frac{1}{2}dr$ and $r+\frac{1}{2}dr$, and vectors from O making angles $\theta-\frac{1}{2}d\theta$ and $\theta+\frac{1}{2}d\theta$ with Ox ; so that its area is

$$\frac{1}{2}\{(r+\frac{1}{2}dr)^2-(r-\frac{1}{2}dr)^2\}d\theta=dr \cdot rd\theta;$$

$$\text{and } A=\iint r dr d\theta, \quad V=\iint z r dr d\theta;$$

the limits of r and θ being taken so as to include all the elements in the area A .

In the general change from the independent variables x, y to any two new variables u, v , the element of area bounded by the curves $u, v, u+du, v+dv$ is ultimately a parallelogram, the coordinates of whose angular points are, relatively to the corner (x, y) ,

$$\frac{\partial x}{\partial u}du, \frac{\partial y}{\partial u}du; \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv, \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv; \frac{\partial x}{\partial v}dv, \frac{\partial y}{\partial v}dv;$$

and therefore the area of this elementary parallelogram is

$$\left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}\right) du dv = \frac{\partial(x, y)}{\partial(u, v)} du dv;$$

so that now, with z expressed as a function of u, v ,

$$A = \iint \frac{\partial(x, y)}{\partial(u, v)} du dv, \quad V = \iint z \frac{\partial(x, y)}{\partial(u, v)} du dv.$$

Thus, for instance, in Thermodynamics, the pressure p and volume v of unit mass of a perfect gas are connected with θ , the absolute temperature, and ϕ , the *entropy*, by the relations (Maxwell, *Heat*)

$$pv = R\theta = (m-n)\theta, \quad p^n v^m = C e^\phi;$$

and, taking θ, ϕ as independent variables,

$$\frac{1}{p} \frac{\partial p}{\partial \theta} + \frac{1}{v} \frac{\partial v}{\partial \theta} = \frac{1}{\theta}, \quad \frac{1}{p} \frac{\partial p}{\partial \phi} + \frac{1}{v} \frac{\partial v}{\partial \phi} = 0;$$

$$\frac{n}{p} \frac{\partial p}{\partial \theta} + \frac{m}{v} \frac{\partial v}{\partial \theta} = 0, \quad \frac{n}{p} \frac{\partial p}{\partial \phi} + \frac{m}{v} \frac{\partial v}{\partial \phi} = 1;$$

so that,
$$\frac{m-n}{p} \frac{\partial p}{\partial \theta} = \frac{m}{\theta}, \quad \frac{m-n}{p} \frac{\partial p}{\partial \phi} = -1;$$

$$\frac{m-n}{v} \frac{\partial v}{\partial \theta} = -\frac{n}{\theta}, \quad \frac{m-n}{v} \frac{\partial v}{\partial \phi} = 1;$$

and
$$\frac{\partial(p, v)}{\partial(\theta, \phi)} = \frac{1}{m-n} \frac{pv}{c\theta} = 1.$$

Therefore the area on the p, v diagram of the *Carnot cycle* bounded by the two *isothermals* θ_1, θ_2 and by the two *adiabatics* ϕ_1, ϕ_2 is $(\theta_1 - \theta_2)(\phi_1 - \phi_2)$.

and from E to D along EPD , dx/dt is negative, so that

$$\int y \frac{dx}{dt} dt = -\text{area } LEPDK;$$

and from D to E along DQE , dx/dt is positive, so that

$$\int y \frac{dx}{dt} dt = \text{area } LEQDK;$$

and therefore, taken round the curve,

$$\int y \frac{dx}{dt} dt, \text{ or } \int y dx = -A;$$

Therefore, taken round the curve,

$$\int \left(y \frac{dx}{dt} + x \frac{dy}{dt} \right) dt, \text{ or } \int (y dx + x dy) = 0;$$

and

$$y dx + x dy = d(xy)$$

is called a *perfect differential*; its integral between two limits is independent of the intermediate values of x and y and of the path described between the limits; so that, taken round any closed path, the integral is zero.

But
$$A = \int \frac{1}{2} \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) dt;$$

and $x dy - y dx$ is *not* a perfect differential, so that its integrated value depends on the path taken between two limits; and in a closed path it represents twice the area enclosed by the path.

Similar theorems hold for the integrals

$$\iint z dx dy \text{ and } \iint z dy dx.$$

138. Changing to polar coordinates by putting

$$x = r \cos \theta, \quad y = r \sin \theta,$$

then

$$x \frac{dy}{dt} - y \frac{dx}{dt} = r^2 \frac{d\theta}{dt};$$

so that

$$A = \int \frac{1}{2} r^2 \frac{d\theta}{dt} dt = \int \frac{1}{2} r^2 d\theta,$$

taken round the curve.

For an origin inside a closed oval curve, the limits of θ may be taken as 0 and 2π .

But if the origin O is outside the area, draw the tangents OF, OH to the curve from O ; then along FPH , $d\theta/dt$ is positive, and

$$\int \frac{1}{2} r^2 \frac{d\theta}{dt} dt = \text{area } OFPCHO;$$

but along HRF , $d\theta/dt$ is negative, and

$$\int \frac{1}{2} r^2 \frac{d\theta}{dt} dt = -\text{area } OFQHO;$$

so that, taken round the curve,

$$A = \int \frac{1}{2} r^2 \frac{d\theta}{dt} dt = \int \frac{1}{2} r^2 d\theta = \int \frac{1}{2} x^2 dv, \text{ if } y/x = v = \tan \theta.$$

Although not a perfect differential, $x dy - y dx$ can be made one by dividing by $x^2 + y^2$, and then

$$\frac{x dy - y dx}{x^2 + y^2} = d \tan^{-1} \frac{y}{x} = d\theta;$$

so that
$$\int \frac{x dy - y dx}{x^2 + y^2} = \int d\theta = 2\pi, \text{ or } 0,$$

according as the origin O is inside or outside the curve.

Consequently if each point of the contour is displaced through small distances $c^2 x/r^2, c^2 y/r^2$, parallel to the axes, the change in the area will be $2\pi c^2$ or zero, according as the origin is inside or outside the contour.

139. A convenient independent variable to take is s , the length of the arc of the perimeter measured from a fixed point; so that the point P may be supposed to move with unit velocity; and now, with p denoting the length of the perpendicular from O on the tangent at P (§ 9),

$$x \frac{dy}{ds} - y \frac{dx}{ds} = p;$$

so that

$$\begin{aligned} A &= \int \frac{1}{2} p ds = \int \frac{1}{2} p \frac{ds}{dr} dr \\ &= \int \frac{1}{2} p \sec \phi dr = \int \frac{1}{2} p r dr / \sqrt{(r^2 - p^2)}. \end{aligned}$$

When fig. 44 represents an indicator diagram, and KL the reduced stroke of the piston, while the ordinate y represents the pressure of the steam, the pencil will describe the contour with the area to the left, when the steam pressure is urging the piston from L to K .

The diagram taken on the return stroke from the other end of the cylinder will be described in the opposite sense, with the area on the right hand of the describing pencil.

Sometimes a loop is found on the diagram, described in the reverse sense; this loop shows that a cushioning effect takes place, which requires attention, as the negative area of this loop represents so much lost work.

In the general case, where the perimeter cuts itself a number of times, then the area obtained by integrating once round a loop will be positive or negative with the above formulas, according as the area is on the left or right of the describing point, as it travels round the curve.

(Clifford, *Common Sense of the Exact Sciences*; Cremona, *Graphical Statics*.)

Familiar instances of such looped contours are seen in Lissajous's figures (Ganot, *Physics*, § 281), whose general equation may be written as (§ 103)

$$m \sin^{-1} x/a - n \sin^{-1} y/b = \text{a constant},$$

$$\text{or} \quad x/a = \sin(nt + \epsilon), \quad y/b = \sin(mt + \epsilon').$$

Also with the polar curves

$$r = a \cos(m\theta/n),$$

where m and n are integers, curves seen on the back of an engine-turned watch.

140. *The Planimeter.*

This instrument in the most usual form, that invented by Amsler of Schaffhausen, consists of two bars OA , AP , pivoted at O and jointed at A , and carrying in PA produced a small graduated roller R , with axis fixed parallel to PA (fig. 45).

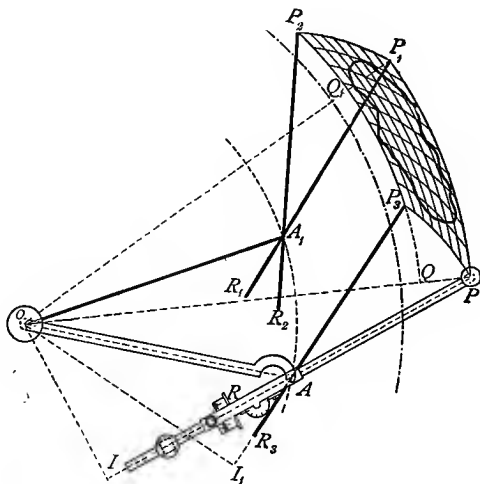


Fig. 45

The instrument is used to measure areas ; the pointer P is carried round the perimeter of the curve whose area is required, an indicator diagram, or the section of a ship for instance ; the roller R , which rolls and slides on the plane of the paper, then registers the area.

If precision is required, the point P may be carried say ten times round the perimeter, and the reading of the roller be divided by ten.

To explain the theory of the instrument, we shall suppose the pointer P to travel round a finite area $PP_1P_2P_3$ (something like a Carnot cycle in Thermodynamics), in which PP_1 , P_2P_3 are circular arcs described round O as centre, and P_1P_2 , P_3P , are arcs round A_1 , A as centres; in this way we analyse the motion due to the joints O and A separately.

Let $OA=a$, $AP=b$, $AR=c$; and let the direction of a positive motion of the roller, as marked by the graduations, be that which on a right-handed screw would give a motion in the direction AR .

Drop the perpendicular OI from O on AR .

(i.) Fix the joint A , and move P to P_1 by rotation round O on the circle PP_1 through an angle θ ; the angular velocity $d\theta/dt$ will give to the roller R the component velocities $OI \cdot d\theta/dt$ in the direction IR and $IR \cdot d\theta/dt$ perpendicular to IR ; the first component drags the roller over the paper, and the second component makes the roller turn with circumferential velocity $IR \cdot d\theta/dt$; and therefore the whole travel of the roller, or its graduations, will be $IR \cdot \theta$.

(ii.) Fix the joint O , and move P_1 to P_2 by rotation round the joint A_1 through an angle ϕ ; the angular velocity $d\phi/dt$ of AP will communicate a circumferential velocity $-cd\phi/dt$ to the roller; and the travel of the roller will therefore be $c\phi$, backwards.

(iii.) Fix the joint A , and move P_2 to P_3 by rotation round O through an angle θ ; the roller will now move backwards, and its travel will be $I_1R \cdot \theta$.

(iv.) Fix the joint O , and move P_3 to P by rotation round A through an angle ϕ ; the travel of the roller will be $c\phi$, forwards; this cancels the travel of (ii.).

In completing the finite circuit $PP_1P_2P_3$, the total forward travel of the roller will then be $(IR - I_1R)\theta$.

But the area $PP_1P_2P_3 = \text{area } PP_1Q_1Q$

$$= \text{sector } OPP_1 - \text{sector } OQQ_1$$

$$= \frac{1}{2}(OP^2 - OP_3^2)\theta$$

$$= \frac{1}{2}(OA^2 + AP^2 + 2AI \cdot AP - OA^2 - AP^2 - 2AI_1 \cdot AP)\theta$$

$$= b(AI - AI_1)\theta = b(IR - I_1R)\theta$$

$$= b \text{ times the travel of the roller.}$$

Thus by altering the length of b by an adjustment of the instrument, which allows the arm AP to slide in the sleeve AR and be clamped, the area can be read off in any required unit, say the square inch or square centimetre.

Any irregular area, such as for instance an indicator diagram or the cross section of a ship, must be supposed built up of infinitesimal elements formed in the same manner as $PP_1P_2P_3$; and will be read off when the pointer P completes a circuit of the perimeter, both joints being now free to turn simultaneously.

When I coincides with R , the roller will not turn, and then P describes a circle called the zero circle, represented by the middle dotted circular line (fig. 45) of radius

$$\sqrt{(OR^2 + RP^2)} = \sqrt{\{a^2 - c^2 + (b+c)^2\}} = \sqrt{(a^2 + b^2 + 2bc)}.$$

When I lies on the same side of R as A , the travel of the roller is reversed, but in a complete circuit the reading is unaltered.

If however the origin O is taken inside the area to be measured, the area of the zero circle must be added to the reading of the roller.

Prof. G. B. Mathews explains the theory of the Planimeter in a slightly different manner, by

(i.) moving OA to OA_1 , and P to P_1 , keeping AP parallel to itself; and then the curvilinear area $APP_1A_1 = b$ times the travel of the roller;

(ii.) moving A_1P_1 to A_1P_2 by rotation round A_1 ;

(iii.) moving OA_1 to OA and P_2 to P_3 , keeping A_1P_2 parallel to itself; and then the curvilinear area $AA_1P_2P_3 = b$ times the travel of the roller, backwards;

(iv.) completing the circuit by moving AP_3 to AP by rotation round A , when the sector $APP_3 = \text{sector } A_1P_1P_2$, and the travel of the roller cancels the travel in step (ii.).

Therefore the

$$\begin{aligned} \text{area } PP_1P_2P_3 &= \text{area } APP_1A_1 - \text{area } A_1P_2P_3A \\ &= b \text{ times the travel of the roller.} \end{aligned}$$

The end A may be guided in a slot of any form and the area will be read off as before; a straight slot is often employed, with the advantage that the pencil P can then cover a greater area; and with appropriate mechanism can be made to register the moment of the area, and its moment of inertia about the straight line of the slot.

(J. F. Bramwell, *British Association*, 1872;

H. S. Hele Shaw, *Proc. I.C.E.*, 1885.)

Examples.

- (1) Prove that the area is $\pi(Bc - bC)$ of the curve, an ellipse (Ex. 4, p. 203), given by

$$x = a + b \cos \theta + c \sin \theta, \quad y = A + B \cos \theta + C \sin \theta.$$

- (2) Trace and find the area of the curve

$$(r - a \cos \theta)^2 = a^2 \cos 2\theta.$$

- (3) Prove that the volume cut off from the surface

$$z^n = Ax^2 + 2Hxy + By^2$$

by the plane $z = c$ is $n/(n+1)$ of the cylinder on the same base.

141. *Functions of three or more Independent Variables.*

A function of three independent variables, x, y, z , denoted by $f(x, y, z)$, may be supposed to represent some function of the position of a point in space whose co-ordinates are x, y, z ; for instance, the density or temperature or pressure at the point.

Then $f(x, y, z) = C$, a constant, would imply a relation connecting x, y, z , and would be the equation of a surface; for instance, a surface of equal density, or pressure.

If V denotes the volume contained in a closed surface S , then

$$V = \iiint dx dy dz,$$

the integration including all the infinitesimal brick shaped elements of volume $dx dy dz$ (fig. 43) contained in S .

When the *density* ρ within the surface S is variable and a given function of x, y, z , then the *mass* M contained by the surface S is given by the triple integration

$$M = \iiint \rho dx dy dz, \text{ so that } \frac{\partial^3 M}{\partial x \partial y \partial z} = \rho,$$

and the mass is the *space integral* of the density ρ throughout the volume V ; while $\bar{x}, \bar{y}, \bar{z}$, the coordinates of the centre of mass, and k_x, k_y, k_z , the radii of gyration about Ox, Oy, Oz , are given by

$$\bar{x}M = \iiint x \rho dx dy dz, \quad \bar{y}M = \iiint y \rho dx dy dz,$$

$$\bar{z}M = \iiint z \rho dx dy dz;$$

$$Mk_x^2 = \iiint (y^2 + z^2) \rho dx dy dz, \quad Mk_y^2 = \iiint (z^2 + x^2) \rho dx dy dz,$$

$$Mk_z^2 = \iiint (x^2 + y^2) \rho dx dy dz.$$

Functions of more than three independent variables cannot be interpreted geometrically without the introduction of the fiction of space of more than three dimensions, a thing which is inconceivable.

A function $F(t, x, y, z)$ of four independent variables t, x, y, z , may however be interpreted, as in Hydrodynamics, as representing the velocity, or density, or pressure, at the time t at a point in space whose coordinates are x, y, z .

142. We have used the words *mass* and *density*; the *mass* of a body is the quantity measured by the balance against certain standard lumps of metal, called *weights* in the Acts of Parliament (French, *poids*, German, *Gewichte*), the standard in this country being the Pound Weight, and in the Metric System the Kilogramme of 1000 grammes.

The *density* of a body is defined as the number of units of mass in the unit of volume; with British units, the density is the number of lbs. in a cubic foot of the substance, and with Metric units is the number of grammes in a cubic centimetre, or of *tonnes* of 1000 kilogrammes per cubic metre.

The units of length are thus the *foot* in the British System, and the *metre* or *centimetre* in the Metric System; while the unit of time in universal use for theoretical investigations is the *second*, the mean solar sexagesimal second.

When therefore we speak of a time t , we mean t seconds; and coordinates x, y, z are measured in the unit of length, which is either the *foot* or else the *metre* or *centimetre*.

For practical purposes there are only three systems of fundamental units which need be considered,

- (i.) the British foot-pound-second (F.P.S.) system;
- (ii.) the C.G.S. (centimetre-gramme-second) system;
- (iii.) the metre-kilogramme-second (M.K.S.) system;

and from these fundamental units of length, mass, and time, all other units, of *area*, *volume*, *density*, *velocity*, *acceleration*, *momentum*, *energy*, *force*, etc., may be derived.

*143. *Spherical Polar Coordinates.*

In this system of coordinates the position of a point P on a sphere with centre at O is defined by θ , its angular distance from a fixed pole N on the sphere, and by ϕ the angle which the plane ONP makes with a fixed initial plane; so that on the terrestrial sphere, ϕ will be the longitude and $\frac{1}{2}\pi - \theta$ the north latitude, if N denotes the north pole.

By taking r the radius of the sphere as variable, we can define the position of any point in space by means of the three quantities r, θ, ϕ , called the spherical polar coordinates in space.

With ON coincident with Oz , and with the plane xOz as the prime meridian, these coordinates are connected with the orthogonal coordinates x, y, z , by means of the relations $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$.

If we replace $r \sin \theta$ by ϖ , then ϖ, ϕ, z , are the coordinates in the cylindrical system (§ 132).

The element of volume cut out by the spheres $r - \frac{1}{2}dr$ and $r + \frac{1}{2}dr$, by the cones $\theta - \frac{1}{2}d\theta$ and $\theta + \frac{1}{2}d\theta$, and by the planes $\phi - \frac{1}{2}d\phi$ and $\phi + \frac{1}{2}d\phi$, will be ultimately

$$dr \cdot r d\theta \cdot r \sin \theta d\phi;$$

so that by triple integration

$$V = \iiint r^2 \sin \theta dr d\theta d\phi, \text{ and } \frac{\partial^3 V}{\partial r \partial \theta \partial \phi} = r^2 \sin \theta;$$

while the mass M and its centre of mass, for variable density ρ , are given by

$$\begin{aligned} M &= \iiint \rho r^2 \sin \theta dr d\theta d\phi, \\ \bar{x}M &= \iiint \rho r^3 \sin^2 \theta \cos \phi dr d\theta d\phi, \\ \bar{y}M &= \iiint \rho r^3 \sin^2 \theta \sin \phi dr d\theta d\phi, \\ \bar{z}M &= \iiint \rho r^3 \sin \theta \cos \theta dr d\theta d\phi. \end{aligned}$$

*144. *Space, Surface, and Line Integrals.*

Consider a fixed closed surface S , and a function X of the coordinates x, y, z of a point in space.

Then in the triple integration extending throughout the volume enclosed by the surface S , called a *space integral*,

$$\iiint \frac{\partial X}{\partial x} dx dy dz = \iint (-X_1 + X_2 - X_3 + \dots) dy dz,$$

where X_1, X_2, X_3, \dots denote the values of X where a point moving from $-\infty$ to ∞ parallel to the axis Ox successively enters and leaves the interior of the surface S .

Denoting by l_1, l_2, l_3, \dots the cosines of the angles which the *outward* drawn normals of the surface S at these points make with Ox , then

$$dy dz = -l_1 dS_1 = l_2 dS_2 = -l_3 dS_3 = \dots,$$

supposing the infinitesimal prism on the base $dy dz$ parallel to Ox to cut out the elements of surface dS_1, dS_2, dS_3, \dots , on entering and leaving the surface S .

Therefore, denoting the element of volume by dV ,

$$\iiint \frac{\partial X}{\partial x} dV = \iint (l_1 X_1 dS_1 + l_2 X_2 dS_2 + \dots) = \iint lX dS \dots (i.),$$

the double integration extending over the surface S , and this is called a *surface integral*; so that a volume integral can always be expressed as a surface integral.

Similarly, with Y, Z other given functions of x, y, z ,

$$\iiint \frac{\partial Y}{\partial y} dV = \iint m Y dS, \quad \iiint \frac{\partial Z}{\partial z} dV = \iint n Z dS;$$

where m, n denote the cosines of the angles the outward drawn normal of the surface S makes with Oy, Oz .

Therefore, by addition

$$\iiint \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) dV = \iint (lX + mY + nZ) dS \dots \dots (ii.).$$

$$\text{Thus } V = \iiint dxdydz = \frac{1}{3} \iint (lx + my + nz) dS.$$

For instance, integrating over the surface of the ellipsoid, p denoting the perpendicular from the centre upon the tangent plane,

$$\int p dS = 4\pi abc, \text{ and } \int dS/p = \frac{4}{3}\pi(bc/a + ca/b + ab/c);$$

and, as an exercise, the student may calculate $\int dS/p$ and $\int dS/p^3$ for the hyperboloids, as well as for the ellipsoid.

Again, suppose X, Y, Z are the component forces per unit of volume acting throughout a fluid at rest, in which the pressure at any point is represented by p ; then the equation of equilibrium of the fluid within the closed surface S , obtained by resolving parallel to Ox , is

$$\iiint X dV = \iint l p dS.$$

But changing the surface integral into a space integral,

$$\iint l p dS = \iiint \frac{\partial p}{\partial x} dV,$$

so that $\frac{\partial p}{\partial x} = X$; and similarly $\frac{\partial p}{\partial y} = Y, \frac{\partial p}{\partial z} = Z$;

or $dp = Xdx + Ydy + Zdz,$

so that the *space variation* of the pressure of a fluid at rest in any direction is equal to the component force per unit of volume in that direction; and surfaces of equal pressure are cut orthogonally by the lines of force.

As another illustration, we may suppose X, Y, Z to represent the components of flux (estimated with British units in lb. per square foot per second) of a fluid in motion; then $\iint (lX + mY + nZ) dS$ represents the number of lb. which is flowing out across the surface S per second; while if ρ denotes the density, in lb. per cubic foot, of the

fluid at any point of the interior of S , so that the mass M within S is $\iiint \rho dV$ lb., then $\partial M/\partial t$ represents the rate of increase, in lb. per second, of the quantity of fluid inside S .

Equating this gain and loss,

$$\partial M/\partial t + \iint (lX + mY + nZ) dS = 0;$$

and, replacing by space integrals,

$$\iiint \left(\frac{\partial \rho}{\partial t} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) dV = 0,$$

leading to the *equation of continuity* in Hydrodynamics, when we replace X, Y, Z by $\rho u, \rho v, \rho w$, where u, v, w denote the components of *velocity* of the fluid.

In a plane, dA denoting an element of the area A , of which ds represents an element of length of the closed contour s , equation (ii.) becomes

$$\iint \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) dA = \int (lX + mY) ds \dots\dots\dots(\text{iii.}),$$

thus expressing a *surface* integral by a *line* integral;

and $l = dy/ds, m = -dx/ds$;

or, if $F(x, y) = 0$ is the equation of the contour,

$$l = \frac{\partial F}{\partial x} / \sqrt{\left\{ \left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 \right\}},$$

$$m = \frac{\partial F}{\partial y} / \sqrt{\left\{ \left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 \right\}}.$$

A similar theorem connects the surface integral on a curved surface S , which is a portion of a closed surface, with a line integral round the edge of S ;

$$\iint \left\{ l \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} \right) + m \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} \right) + n \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) \right\} dS$$

$$= \int \left(X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds} \right) ds \dots\dots\dots(\text{iv.})$$

(Maxwell, *Electricity*, i., p. 25), this surface integral vanishing, by equation (ii.), for a closed surface.

For a plane surface, we may take $l=0, m=0, n=1$; and now, as in (iii.),

$$\iint \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dA = \int \left(X \frac{dx}{ds} + Y \frac{dy}{ds} \right) ds.$$

*145. *Green's Theorem.*

Now suppose U and U' are given functions of x, y, z ; then from equation (i.), integrating by parts,

$$\iiint \frac{\partial U}{\partial x} \frac{\partial U'}{\partial x} dV = \iint l U' \frac{\partial U}{\partial x} dS - \iiint U' \frac{\partial^2 U}{\partial x^2} dV \dots (\text{v.});$$

and therefore, denoting by $-\nabla^2$ the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

$$\begin{aligned} & \iiint \left(\frac{\partial U}{\partial x} \frac{\partial U'}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial U'}{\partial y} + \frac{\partial U}{\partial z} \frac{\partial U'}{\partial z} \right) dV \\ &= \iint U' \left(l \frac{\partial U}{\partial x} + m \frac{\partial U}{\partial y} + n \frac{\partial U}{\partial z} \right) dS + \iiint U' \nabla^2 U dV \\ &= \iint U' \frac{\partial U}{\partial \nu} dS + \iiint U' \Delta^2 U dV, \dots \dots \dots (\text{vi.}), \end{aligned}$$

and therefore, by symmetry,

$$= \iint U \frac{\partial U'}{\partial \nu} dS + \iiint U \Delta^2 U' dV, \dots \dots \dots (\text{vii.}),$$

where
$$l \frac{\partial U}{\partial x} + m \frac{\partial U}{\partial y} + n \frac{\partial U}{\partial z} = \frac{\partial U}{\partial \nu},$$

representing the rate of growth of U in the direction of the outward drawn normal of S .

Equations (vi.) and (vii.) constitute *Green's Theorem*, a theorem of great use in the mathematical theories of Electricity and Magnetism.

(*An Essay on Electricity and Magnetism*, by G. Green; edited by N. M. Ferrers.)

*146. *Change of the Variables in Space Integrals.*

Generally in changing from x, y, z to any new independent variables u, v, w , we may consider that space is divided up into elements of volume bounded by the surfaces for which u, v, w are constant; and now the element of volume will be changed from $dx dy dz$ to

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} du dv dw,$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} \text{ denoting the determinant } \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

called the Jacobian of x, y, z with respect to u, v, w .

Denoting by ds , the element of length, then ds^2 becomes changed from $dx^2 + dy^2 + dz^2$ to

$$\left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw \right)^2 + \left(\frac{\partial y}{\partial u} du + \dots \right)^2 + \left(\frac{\partial z}{\partial u} du + \dots \right)^2$$

$$= A^2 du^2 + B^2 dv^2 + C^2 dw^2 + 2D dv dw + 2E dw du + 2F du dv,$$

$$A^2 = \frac{\partial x^2}{\partial u^2} + \frac{\partial y^2}{\partial u^2} + \frac{\partial z^2}{\partial u^2}, \dots, D = \frac{\partial x}{\partial v} \frac{\partial x}{\partial w} + \frac{\partial y}{\partial v} \frac{\partial y}{\partial w} + \frac{\partial z}{\partial v} \frac{\partial z}{\partial w}, \dots$$

It is convenient to choose the new variables u, v, w , so that the corresponding surfaces cut at right angles, and then D, E, F vanish; while

$$\frac{1}{A} \frac{\partial x}{\partial u}, \frac{1}{A} \frac{\partial y}{\partial u}, \frac{1}{A} \frac{\partial z}{\partial u}$$

represent the cosines of the angles the normal to the surface u makes with the axes Ox, Oy, Oz .

But with x, y, z as independent variables, and denoting

$$\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} + \frac{\partial u^2}{\partial z^2} \text{ by } h_1^2, \frac{\partial v^2}{\partial x^2} + \dots \text{ by } h_2^2, \frac{\partial w^2}{\partial x^2} + \dots \text{ by } h_3^2;$$

then
$$\frac{1}{h_1} \frac{\partial u}{\partial x}, \frac{1}{h_1} \frac{\partial u}{\partial y}, \frac{1}{h_1} \frac{\partial u}{\partial z}$$

are also the cosines of the angles which the normal to the surface u makes with Ox, Oy, Oz ; so that

$$\frac{1}{h_1} \frac{\partial u}{\partial x} = \frac{1}{A} \frac{\partial x}{\partial u}, \frac{1}{h_1} \frac{\partial u}{\partial y} = \frac{1}{A} \frac{\partial y}{\partial u}, \frac{1}{h_1} \frac{\partial u}{\partial z} = \frac{1}{A} \frac{\partial z}{\partial u},$$

and similarly,

$$\begin{aligned} \frac{1}{h_2} \frac{\partial v}{\partial x} &= \frac{1}{B} \frac{\partial x}{\partial v}, \frac{1}{h_2} \frac{\partial v}{\partial y} = \frac{1}{B} \frac{\partial y}{\partial v}, \frac{1}{h_2} \frac{\partial v}{\partial z} = \frac{1}{B} \frac{\partial z}{\partial v}, \\ \frac{1}{h_3} \frac{\partial w}{\partial x} &= \frac{1}{C} \frac{\partial x}{\partial w}, \frac{1}{h_3} \frac{\partial w}{\partial y} = \frac{1}{C} \frac{\partial y}{\partial w}, \frac{1}{h_3} \frac{\partial w}{\partial z} = \frac{1}{C} \frac{\partial z}{\partial w}, \end{aligned}$$

Also, since
$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz,$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz,$$

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz;$$

therefore, multiplying these equations by

$$\frac{1}{h_1^2} \frac{\partial u}{\partial x}, \frac{1}{h_2^2} \frac{\partial v}{\partial x}, \frac{1}{h_3^2} \frac{\partial w}{\partial x},$$

and adding,
$$dx = \frac{1}{h_1^2} \frac{\partial u}{\partial x} du + \frac{1}{h_2^2} \frac{\partial v}{\partial x} dv + \frac{1}{h_3^2} \frac{\partial w}{\partial x} dw;$$

so that
$$\frac{\partial x}{\partial u} = \frac{1}{h_1^2} \frac{\partial u}{\partial x}, \dots; \text{ or } Ah_1 = Bh_2 = Ch_3 = 1.$$

Now denoting by ds_1, ds_2, ds_3 , the elements of the normals, intersections of the surfaces u, v, w ,

$$ds_1 = Adu, ds_2 = Bdv, ds_3 = Cdw;$$

so that
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = ABC, \text{ and } \frac{\partial(u, v, w)}{\partial(x, y, z)} = h_1 h_2 h_3 = 1/ABC$$

Now if V denotes any function of x, y, z , or u, v, w , then

$$\frac{\partial V^2}{\partial x^2} + \frac{\partial V^2}{\partial y^2} + \frac{\partial V^2}{\partial z^2} = h_1^2 \frac{\partial V^2}{\partial u^2} + h_2^2 \frac{\partial V^2}{\partial v^2} + h_3^2 \frac{\partial V^2}{\partial w^2};$$

also,

$$-\nabla^2 V = h_1 h_2 h_3 \left\{ \frac{\partial}{\partial u} \left(\frac{h_1}{h_2 h_3} \frac{\partial V}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_2}{h_3 h_1} \frac{\partial V}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_3}{h_1 h_2} \frac{\partial V}{\partial w} \right) \right\};$$

which are readily proved by taking the axes parallel to the normals to u, v, w at the point.

With spherical polar coordinates r, θ, ϕ (§ 144),

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin \theta d\phi^2,$$

so that $A = 1, B = r, C = r \sin \theta$;

$$-\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{r^2 \partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}.$$

*147. Confocal Quadrics.

A familiar instance occurs with confocal quadric surfaces, where, with λ, μ, ν for new variables,

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1, \quad \frac{x^2}{a^2 + \mu} + \dots = 1, \quad \frac{x^2}{a^2 + \nu} + \dots = 1;$$

whence, by solution,

$$x^2 = \frac{(a^2 + \lambda)(a^2 + \mu)(a^2 + \nu)}{(a^2 - b^2)(a^2 - c^2)}, \quad y^2 = \dots, \quad z^2 = \dots;$$

and
$$A^2 = \frac{(\lambda - \mu)(\lambda - \nu)}{4(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}, \quad B^2 = \dots, \quad C^2 = \dots$$

Denoting by p the perpendicular from the centre on the tangent plane of the surface λ , then we can show that

$$\frac{1}{p^2} = \frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2}{(c^2 + \lambda)^2}, \quad \frac{\partial \lambda}{\partial x} = \frac{2p^2 x}{a^2 + \lambda}, \quad \dots, \quad h_1^2 = 4p^2,$$

and
$$\frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2} + \frac{\partial^2 \lambda}{\partial z^2} = 2p^2 \left(\frac{1}{a^2 + \lambda} + \frac{1}{b^2 + \lambda} + \frac{1}{c^2 + \lambda} \right);$$

and thence that $V = f\lambda$ will satisfy the condition $\nabla^2 V = 0$,

provided $f\lambda = C \int \{ (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda) \}^{-\frac{1}{2}} d\lambda$.

Examples.

- (1) Interpret geometrically the equations

$$fr=0, f\theta=0, f\phi=0, f(\theta, \phi)=0, F(r, \theta)=0, F(r, \phi)=0.$$

- (2) Prove that the quadric surfaces of revolution,

$$\frac{x^2+y^2}{\sec^2 u} + \frac{z^2}{\tan^2 u} = c^2 \dots\dots\dots (i.),$$

$$\frac{x^2+y^2}{\operatorname{sech}^2 v} - \frac{z^2}{\tanh^2 v} = c^2 \dots\dots\dots (ii.),$$

$$\frac{x^2}{\cos^2 w} - \frac{y^2}{\sin^2 w} = 0 \dots\dots\dots (iii.),$$

are (i.) oblate spheroids, (ii.) hyperboloids of one sheet, (iii.) planes, forming a system of confocal orthogonal surfaces; and sketch the figure.

Prove that they intersect in the point

$$x = c \sec u \operatorname{sech} v \cos w, \quad y = c \sec u \operatorname{sech} v \sin w, \\ z = c \tan u \tanh v;$$

and that the generating lines of (ii.) are given by

$$\frac{x - c \sec u \operatorname{sech} v \cos w}{\operatorname{sech} v \sin(u \pm w)} = \frac{y - c \sec u \operatorname{sech} v \sin w}{-\operatorname{sech} v \cos(u \pm w)} = \frac{z - c \tan u \tanh v}{\pm \tanh v};$$

the angle between them being $2\cos^{-1}(\cos u \operatorname{sech} v)$.

Write down the corresponding equations for a system of confocal prolate spheroids and hyperboloids of two sheets.

- (3) Prove that the equations

$$\frac{y^2}{\cosh^2 \frac{1}{2} u} + \frac{z^2}{\sinh^2 \frac{1}{2} u} = 8a(a \cosh u - x) \dots\dots\dots (i.),$$

$$\frac{y^2}{\cos^2 \frac{1}{2} v} - \frac{z^2}{\sin^2 \frac{1}{2} v} = 8a(a \cosh v - x) \dots\dots\dots (ii.),$$

$$\frac{y^2}{\sinh^2 \frac{1}{2} w} + \frac{z^2}{\cosh^2 \frac{1}{2} w} = 8a(a \cosh w + x) \dots\dots\dots (iii.)$$

represents a system of confocal paraboloids intersecting at right angles in the point

$$x = a(\cosh u + \cos v - \cosh w),$$

$$y = 4a \cosh \frac{1}{2}u \cos \frac{1}{2}v \sinh \frac{1}{2}w, \quad z = 4a \sinh \frac{1}{2}u \sin \frac{1}{2}v \cosh \frac{1}{2}w.$$

Determine A, B, C for this system, and the equations of the generating lines of (ii).

Prove also that $\nabla^2(u, v, \text{ or } w) = 0$; as also in Ex. 2.

- (4) Verify that $1/r, \phi, \log \tan \frac{1}{2}\theta$, and their products are annihilated by the operator ∇^2 .

Also $r^{-2}\cos \theta, r^{-2}\sin \theta \cos(\phi + \alpha)$.

- (5) Prove that, if A and B are the ends of a diameter of a sphere of radius a whose centre is at O , the function

$$V = \frac{1}{AP} - \frac{1}{BP} - \frac{1}{AB} \log \frac{AN + AP}{BN + BP},$$

where N is the foot of the perpendicular from P on AB , is annihilated by ∇^2 ; and also that $\partial V / \partial r = 0$, when $r = a$.

Give the physical interpretation of this result.

- (6) Prove that

$$\begin{aligned} & \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2} + \dots + \frac{\partial^2 V}{\partial x_n^2} \\ &= \frac{1}{W} \left\{ \frac{\partial}{\partial r} \left(W \frac{\partial V}{\partial r} \right) + \frac{\partial}{\partial \theta_1} \left(\frac{W}{u_1} \frac{\partial V}{\partial \theta_1} \right) + \dots + \frac{\partial}{\partial \theta_{n-1}} \left(\frac{W}{u_{n-1}} \frac{\partial V}{\partial \theta_{n-1}} \right) \right\}, \end{aligned}$$

where

$$x_1 = r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-1}, \quad u_1 = r^2,$$

$$x_2 = r \sin \theta_1 \sin \theta_2 \dots \cos \theta_{n-1}, \quad u_2 = r^2 \sin^2 \theta_1,$$

$$x_3 = r \sin \theta_1 \sin \theta_2 \dots \cos \theta_{n-2}, \quad u_3 = r^2 \sin^2 \theta_1 \sin^2 \theta_2,$$

.....

$$x_{n-1} = r \sin \theta_1 \cos \theta_2, \quad u_{n-1} = r^2 \sin^2 \theta_1 \sin^2 \theta_2 \dots \sin^2 \theta_{n-2},$$

$$x_n = r \cos \theta_1, \quad W = r^{n-1} \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2}.$$

Examine the case of $n = 3$.

Prove also that

$$x_1 \frac{\partial V}{\partial x_1} + x_2 \frac{\partial V}{\partial x_2} + \dots + x_n \frac{\partial V}{\partial x_n} = r \frac{\partial V}{\partial r}.$$

148. *Quantics.*

A rational integral *homogeneous* algebraical function of the n^{th} degree in m variables x, y, z, \dots is defined to be "a function in which the sum of the indices of the variables in each term is constant and equal to n , the indices being positive integers"; such a function is called a *quantic*, and is denoted by $(x, y, z, \dots)^n$.

Thus $(x, y)^n$, or $(a, b, c, \dots)(x, y)^n$ represents the *binary* quantic in x, y of the n^{th} degree

$$ax^n + nbx^{n-1}y + \frac{1}{2}n(n-1)cx^{n-2}y^2 + \dots$$

Denoting the general quantic by u , then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + \dots = nu.$$

This is proved for each single term of the quantic, say

$$Ax^p y^q z^r \dots, \text{ where } p + q + r + \dots = n;$$

for, denoting this term by v , then

$$x \frac{\partial v}{\partial x} = pv, \quad y \frac{\partial v}{\partial y} = qv, \quad z \frac{\partial v}{\partial z} = rv, \dots;$$

and

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} + z \frac{\partial v}{\partial z} + \dots = nv.$$

More generally

$$\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + \dots \right)^2 u = n(n-1)u,$$

$$\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + \dots \right)^3 u = n(n-1)(n-2)u,$$

and so on; in which it is important to notice that the expression in brackets on the left hand side must be ex-

panded by the Multinomial Theorem as if $x, \frac{\partial}{\partial x}, y, \frac{\partial}{\partial y}, \dots$ were independent algebraical quantities, and then u supplied in the partial derivatives.

For if $\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \dots\right)^k u$ meant that the operation $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \dots$ was repeated k times, we should have

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \dots\right)^k u = n^k u.$$

The operator is therefore a particular case of the operator $x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \dots$, denoted by Δ , and called by Klein the *polarizing operator*.

These theorems are proved by expanding the quantic

$$u_1 = (x + hx, y + hy, z + hz, \dots)^n = (1 + h)^n u$$

in the form (§ 126)

$$\begin{aligned} u_1 = u + h\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \dots\right)u \\ + \frac{h^2}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \dots\right)^2 u + \dots, \end{aligned}$$

and equating coefficients of like powers of h .

These theorems are called *Euler's Theorems of Homogeneous Functions, or Quantics*.

A non-homogeneous function can be always made to appear homogeneous by the introduction of an appropriate factor to each term, consisting of the requisite power of some new variable, which may afterwards be interpreted as unity.

Thus the general equation of a conic section

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

can be made homogeneous, and a *ternary quadratic* in x, y, z , by writing it

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0;$$

and now x, y, z may be considered as trilinear coordinates; and replacing z by unity gives the ordinary Cartesian coordinates, with oblique axes.

*149. *Arbogast's Method of Derivation.*

Denoting $f(x+x't, y+y't, z+z't, \dots)$ by f , then

$$\frac{df}{dt} = x' \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + z' \frac{\partial f}{\partial z} + \dots;$$

so that the polarizations of $f(x, y, z, \dots)$ by the successive operations of Δ are the coefficients of $t, t^2/2!, t^3/3!, \dots$ in the expansion of f in powers of t .

Again, if we denote

$$a_0 + a_1x + a_2 \frac{x^2}{2!} + \dots + a_r \frac{x^r}{r!} + \dots \text{ by } y,$$

$$\begin{aligned} \text{then} \quad \frac{dFy}{dx} &= \left(a_1 + a_2x + \dots + a_{r+1} \frac{x^r}{r!} + \dots \right) F'y \\ &= \left(a_1 \frac{\partial}{\partial a_0} + a_2 \frac{\partial}{\partial a_1} + \dots + a_{n+1} \frac{\partial}{\partial a_n} + \dots \right) Fy; \end{aligned}$$

and thence generally

$$\frac{d^n Fy}{dx^n} = \left(a_1 \frac{\partial}{\partial a_0} + a_2 \frac{\partial}{\partial a_1} + \dots + a_{r+1} \frac{\partial}{\partial a_r} + \dots \right)^n Fy.$$

Therefore, expanded in powers of x by Maclaurin's Theorem,

$$Fy = \sum \frac{x^n}{n!} \left(a_1 \frac{\partial}{\partial a_0} + a_2 \frac{\partial}{\partial a_1} + \dots + a_{r+1} \frac{\partial}{\partial a_r} + \dots \right)^n fa_0;$$

so that the coefficients of the powers of x are derived by the repeated operation of

$$a_1 \frac{\partial}{\partial a_0} + a_2 \frac{\partial}{\partial a_1} + \dots + a_{r+1} \frac{\partial}{\partial a_r} + \dots \text{ on } fa_0;$$

this process is called *Arbogast's Method of Derivation*.

In Arbogast's second method of Derivation

$$fy = f(a + bx + cx^2 + dx^3 + ex^4 + \dots),$$

is expanded in powers of x , in the form

$$\begin{aligned} fy &= \sum \frac{x^n}{n!} \left(b \frac{\partial}{\partial a} + 2c \frac{\partial}{\partial b} + 3d \frac{\partial}{\partial c} + 4e \frac{\partial}{\partial d} + \dots \right)^n fa, \\ &= A + Bx + Cx^2 + Dx^3 + Ex^4 + \dots, \end{aligned}$$

suppose; and now we find

$$A = fa$$

$$B = f'a \cdot b$$

$$C = f'a \cdot c + f''a \cdot \frac{b^2}{2!}$$

$$D = f'a \cdot d + f''a \cdot bc + f'''a \cdot \frac{b^3}{3!}$$

$$E = f'a \cdot e + f''a(bd + \frac{1}{2}c^2) + f'''a \cdot \frac{1}{2}b^2c + f''''a \cdot \frac{b^4}{4!}$$

.....

and so on, by a simpler mode of Derivation.

For example, applying this method to Ex. 3, p. 234,

$$\sqrt{1+x} = \exp(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{4}x^3 + \frac{1}{6}x^4 \dots);$$

here $a = 1, b = -\frac{1}{2}, c = \frac{1}{8}, d = -\frac{1}{4}, e = \frac{1}{6}, \dots,$

and $fa = f'a = f''a = \dots = e;$

thence $A = 1, B = -\frac{1}{2}, C = \frac{1}{2} \cdot \frac{1}{4}, D = -\frac{7}{16}, E = \frac{2}{5} \cdot \frac{4}{6} \cdot \frac{7}{6}, \dots$

*150. *Theorems of Lagrange, Laplace, and Burmann.*

Put $y = a + bx + cx^2 + dx^3 + \dots,$

and $\phi y = b + cx + dx^2 + ex^3 + \dots;$

so that $y = a + x\phi y.$

Then $\frac{\partial y}{\partial x} = \phi y + x\phi'y \frac{\partial y}{\partial x}, \frac{\partial y}{\partial a} = 1 + x\phi'y \frac{\partial y}{\partial a};$

and thus $\frac{\partial y}{\partial x} = \phi y \frac{\partial y}{\partial a}.$

Then if $u = fy,$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \frac{\partial y}{\partial x} = \phi y f'y \frac{\partial y}{\partial a} = \phi y \frac{\partial u}{\partial a};$$

and since

$$\frac{\partial}{\partial x} \left(Fy \frac{\partial y}{\partial a} \right) = F'y \frac{\partial y}{\partial x} \frac{\partial y}{\partial a} + Fy \frac{\partial^2 y}{\partial x \partial a} = \frac{\partial}{\partial a} \left(Fy \frac{\partial y}{\partial x} \right),$$

therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\phi y f'y \frac{\partial y}{\partial a} \right) = \frac{\partial}{\partial a} \left(\phi y f'y \frac{\partial y}{\partial x} \right) = \frac{\partial}{\partial a} \left\{ (\phi y)^2 \frac{\partial u}{\partial a} \right\};$$

and generally, by Induction,

$$\frac{\partial^n u}{\partial x^n} = \frac{\partial^{n-1}}{\partial a^{n-1}} \left\{ (\phi y)^n \frac{\partial u}{\partial a} \right\}.$$

Now make $x=0$, therefore $y=a$, and $u=fa$; and

$$u = fy = fa + \sum \frac{x^n}{n!} \frac{\partial^{n-1}}{\partial a^{n-1}} \{ (\phi a)^n f'a \};$$

this is called *Lagrange's Theorem*.

Thus, for instance, putting $fy = \phi y$, we find

$$b = \phi a, c = \frac{1}{2!} \frac{db^2}{da}, d = \frac{1}{3!} \frac{d^2 b^3}{da^2}, \dots$$

Again, suppose $y = \mu + \frac{1}{2}h(y^2 - 1)$;

then
$$y = \mu + \sum \frac{h^n}{n!} \frac{d^{n-1}}{d\mu^{n-1}} \left(\frac{\mu^2 - 1}{2} \right)^n$$

But, on the condition that $y = \mu$ when $h = 0$,

$$y = \frac{1}{h} \{ 1 - \sqrt{(1 - 2\mu h + h^2)} \}, \quad \frac{\partial y}{\partial \mu} = (1 - 2\mu h + h^2)^{-\frac{1}{2}};$$

so that (ex. 6, p. 276)

$$P_n = \frac{1}{n!} \frac{d^n}{d\mu^n} \left(\frac{\mu^2 - 1}{2} \right)^n.$$

Laplace has given a slightly more general form to the theorem of Lagrange by putting

$$z = Fy = F(a + x\phi y);$$

and now $\frac{\partial z}{\partial x} = \phi y \frac{\partial z}{\partial a}$, as before; and

$$fz = f(Fa) + \sum \frac{x^n}{n!} \frac{\partial^{n-1}}{\partial a^{n-1}} [\{ \phi(Fa) \}^n f'(Fa)].$$

Next put x or $(y-a)/\phi y = \psi y$; then

$$fy = fa + \sum \frac{(\psi y)^n}{n!} \frac{\partial^{n-1}}{\partial y^{n-1}} \left\{ \frac{(y-a)^n f'y}{(\psi y)^n} \right\}_{(y=a)};$$

called *Burmah's Theorem*, giving the expansion of fy in powers of any other function ψy , $y=a$ being a root of $\psi y = 0$.

CHAPTER VI.

CURVES IN GENERAL.

151. *Curve Tracing in Cartesian Coordinates.*

A number of such curves have already been introduced previously, which presented no difficulty in tracing from their equations; a slight sketch will now be given of a systematic method of treatment, but for a more complete account the student is referred to the treatises on *Curve Tracing* by Frost or Woolsey Johnson.

Given the equation of a curve in the rational integral algebraical form of the implicit relation $f(x, y)=0$, first group the terms in *binary quantics* (§ 148), arranged in descending degree $n, n-1, \dots, 3, 2, 1, 0$; and denote them by $u_n, u_{n-1}, \dots, u_3, u_2, u_1, u_0$; so that the equation of the curve becomes

$$u_n + u_{n-1} + \dots + u_3 + u_2 + u_1 + u_0 = 0; \dots\dots\dots (A)$$

then n is called the *degree* of the curve.

If u_0 does not vanish, the curve does not pass through the origin; but by changing the origin to a point on the curve we can make u_0 vanish, and now the new u_1 , equated to zero, gives the tangent at this point.

Also $u_2 + u_1 = 0$ is the equation of a conic section, osculating or having a contact of the second order (§ 118) at the point.

With rectangular axes, $C(x^2 + y^2) + u_1 = 0$ will be the equation of a circle touching the curve; and if

$$u_1 = Ax + By, \quad u_2 = ax^2 + 2hxy + by^2,$$

then this circle will be the circle of curvature, if

$$C = (aB^2 - 2hAB + bA^2)/(A^2 + B^2).$$

Similarly $u_3 + u_2 + u_1 = 0$ will represent a *cubic* curve, having a contact of the *third* order (§ 118), and so on.

If u_1 also disappears, then $u_2 = 0$ represents two straight lines, real or imaginary, through the origin, which are tangents at this point; if they are real, the origin is a *double point*, and the curve crosses itself; if imaginary, the origin is a *conjugate point*, that is, an isolated point, the coordinates of which satisfy the equation of the curve.

If u_2 is a perfect square, the tangents are coincident and the origin is in general a *cusp* (§ 104).

If u_2 also vanishes, then $u_3 = 0$ denotes the tangents at the origin; so that if u_3 has three real linear factors, the origin is a *triple point*; and so on.

Generally to find the *multiple points* of a curve, that is the points where the curve crosses itself, consider the first derived equation of (A) (§ 125)

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0;$$

this gives in general a determinate value of dy/dx ; except when $\partial f/\partial x = 0$, and $\partial f/\partial y = 0$, when the value of dy/dx becomes *indeterminate*, and the second, or third, ... derived equation must be employed to determine dy/dx ; so that to determine the *multiple points* of a curve, we must find the values of x and y from the equations $\partial f/\partial x = 0$, $\partial f/\partial y = 0$, which also satisfy the equation of the curve $f(x, y) = 0$; and then a change of origin to one of these points will indicate its nature by inspection.

Newton's Analytical Parallelogram.

In tracing a curve in the neighbourhood of one of these singular points, taken as origin, it is important to know which terms in the equation can be neglected in comparison with a pair of others.

In Newton's Parallelogram a term of the equation Ax^my^n is represented graphically by a letter A at the point whose coordinates are m, n on a diagram; and now if another term Bx^py^q is placed on the diagram, the line AB will divide the diagram into two parts, such that a third term Cx^ry^s can be rejected if its representative point C lies on the side of AB remote from the origin, as being an infinitesimal of a higher order than the terms A and B , which have been retained.

When all the other terms of the equation can be rejected in comparison with Ax^my^n and Bx^py^q , then

$$Ax^my^n + Bx^py^q = 0, \text{ or } Ax^{m-p} + By^{q-n} = 0$$

will give a close approximation to the curve.

Examples.

(1) Determine the tangents at the origin of

- (i.) $x^2(x+y) - a^2(x-y) = 0$;
 - (ii.) $x^2y^2 - a^3(x+y) = 0$;
 - (iii.) $x^2y^2 - a^2(x^2 - y^2) = 0$;
 - (iv.) $x^4 + y^4 + 6ax^2y - 8ay^3 = 0$;
 - (v.) $x^4 + 3ax^2y + 2axy^2 - ay^3 = 0$;
 - (vi.) $x^2 + y^2 = a^2, 2ax, 2ay$;
 - (vii.) $x^3 + y^3 = a^3, a^2x, a^2y, ax^2, 3axy, ay^2$;
 - (viii.) $x^4 + y^4 = a^4, a^3x, \dots, a^2x^2, a^2xy, ax^3, ax^2y$;
 - (ix.) $x^5 + y^5 = a^5, a^4x, a^3x^2, a^3xy, a^2x^3, ax^4, ax^3y, ax^2y^2, \dots$;
- and sketch the curves near O , by means of Newton's Parallelogram.

- (2) Determine the equation of the tangent of a circle, parabola, ellipse, hyperbola, and generally of a conic at any point, by changing the origin to the point and back again.
- (3) Prove that chords of a conic, which subtend a right angle at a fixed point O in the conic, pass through a fixed point on the normal at O .

152. *Asymptotes.*

To determine the nature of the curve at an infinite distance from the origin, consider the geometrical interpretation of the equation $u_n=0$, which represents n straight lines, real or imaginary, through the origin.

The real straight lines will approximate to the nature of the curve at infinity, and will be parallel to the rectilinear *asymptotes*, if asymptotes exist.

A rectilinear *asymptote* is defined to be "a straight line, at a finite distance from the origin, to which a branch of the curve continually approaches, and ultimately at an infinite distance becomes indefinitely near."

The equation of an asymptote will therefore be of the form $y=mx+c$, when $y=mx$ is the equation of one of the straight lines represented by $u_n=0$; the asymptotes of a hyperbola are familiar instances.

The problem of finding an asymptote is then, from the implicit relation $f(x, y)=0$, to expand y by *reversion of series* (§ 85) in descending powers of x , in the form

$$y=mx+c+px^{-1}+qx^{-2}+\dots$$

Substituting this expansion of y in terms of x in the equation $f(x, y)=0$, and treating the resulting equation as an identity, then by equating to zero the coefficients

of $x^n, x^{n-1}, x^{n-2}, \dots$, sufficient equations are obtained to determine m, c, p, q, \dots .

Then m determines the direction and c the position of the asymptote, while p , or in its absence q, \dots determines the side of the asymptote on which the curve lies; for this reason it is generally useful to expand y in descending powers of x as far as three terms.

We notice that if u_{n-1} is absent, and $y - mx$ is not a repeated factor of u_n , then $c=0$, and the corresponding asymptote passes through the origin.

Thus, the general equation of a conic section, of eccentricity e , and semi-latus-rectum l , is

$$y^2 = 2lx + (e^2 - 1)x^2,$$

when the origin is at a vertex; and putting

$$y = mx + c + px^{-1} + \dots,$$

we find $m^2 = e^2 - 1, c = l/m, p = -n^2/2m, \dots$

The asymptotes are therefore real only for hyperbolas, in which $e > 1$; and their equation will then be

$$y = \pm \sqrt{(e^2 - 1)} \left(x + \frac{l}{e^2 - 1} \right).$$

When $e=1$, the conic is a parabola, and the asymptote is given by $m=0, c=\infty$; so that it lies at an infinite distance, and does not therefore satisfy the definition of an asymptote.

If u_n has a factor x , then to determine the corresponding asymptote, we must expand x in descending powers of y , in the form

$$x = c' + p'y^{-1} + q'y^{-2} + \dots;$$

or we may put, in general,

$$x = m'y + c' + p'y^{-1} + \dots,$$

and determine m', c', p', \dots as before.

Sometimes the expansion of y in descending powers of x or x in powers of y must be written

$$y = Ax^2 + mx + c + px^{-1} + \dots,$$

$$x = A'y^2 + m'y + c' + p'y^{-1} + \dots,$$

and then $y = Ax^2 + mx + c$, or $x = A'y^2 + m'y + c'$ is called a *parabolic asymptote*; and so on.

When it is possible to obtain y *explicitly* in terms of x , or x in terms of y (§ 13) from the *implicit* relation $f(x, y) = 0$, the asymptotes are then readily determined by expanding by the Binomial Theorem and other algebraical operations in descending powers.

Thus if $x^3 + y^3 = a^3$,

then $y = \sqrt[3]{(a^3 - x^3)} = -x(1 - a^3x^{-3})^{\frac{1}{3}} = -x + \frac{1}{3}a^3x^{-2} + \dots$;

or $x = \sqrt[3]{(a^3 - y^3)} = -y(1 - a^3y^{-3})^{\frac{1}{3}} = -y + \frac{1}{3}a^3y^{-2} + \dots$.

Also if $x = a$ makes $y = \infty$, or $y = b$ makes $x = \infty$, then $x - a = 0$ is an asymptote; and so also is $y - b = 0$.

For instance, if the equation of the curve is

$$(a/x)^2 + (b/y)^2 = 1,$$

then $y^2 = b^2x^2/(x^2 - a^2)$, and $x^2 = a^2y^2/(y^2 - b^2)$;

so that $x \pm a = 0$, and $y \pm b = 0$ are asymptotes.

The preceding considerations are in general sufficient for tracing a curve whose Cartesian equation is given, but considerations of symmetry are also useful; thus if *even* powers only of x appear in the equation, the curve is symmetrical right and left of the axis of y : if *even* powers only of y appear, the curve is symmetrical above and below the axis of x , as if reflected in Ox .

If x and y are involved symmetrically, then $x - y = 0$ is an axis of symmetry, in which the curve is reflected, and on being doubled along this line the two halves of the curve will come into coincidence.

Examples.

- (1) Determine the asymptotes of the following curves, and draw them:—

- (i.) $x^2 - y^2 = a^2$; (ii.) $(x/a)^2 - (y/b)^2 = 1$;
 (iii.) $y^2 = 2ax + x^2$; (iv.) $x(x^2 + y^2) - ay^2 = 0$;
 (v.) $x^2y + xy^2 = a^3$; (vi.) $x^3 - xy^2 + ay^2 = 0$;
 (vii.) $x^2y^2 = a^2(x^2 + y^2)$, or $a^2(x^2 - y^2)$;
 (viii.) $x^4 - y^4 - a^2xy = 0$; (ix.) $xy(x^2 - y^2) = a^4$;
 (x.) $xy(x^2 + y^2) = a(x^3 + y^3)$;
 (xi.) $\alpha\beta\gamma = c^3$, where α, β, γ are the perpendiculars on the sides of an equilateral triangle.

- (2) Determine the asymptotes of the curves in the preceding set of examples (p. 317), and draw them.

- (3) Determine the equation of the cubic curve which has the asymptotes

$$2x - y + 3a = 0, x + y - 3a = 0, x + y + a = 0,$$

and cuts the axis of x at O at an angle $\tan^{-1}(-2)$.

- (4) Draw, for different values of c , the curves

- (i.) $x^2 + 2cxy + y^2 = 1$;
 (ii.) $x^2 + 2cxy + x^2 = 1 - c^2$,

equivalent to

$$c = xy + \sqrt{(1 - x^2)}\sqrt{(1 - y^2)}, \text{ or } xy + \sqrt{(x^2 - 1)}\sqrt{(y^2 - 1)};$$

- (iii.) $x^4 + 2cx^2y^2 + y^4 = 1$;
 (iv.) $x^4 + 2cx^2y^2 + y^4 = 1 - c^2$.

- (5) Draw the curve $(x^2 - a^2)^2 + (y^2 - b^2)^2 = c^4$,

- (i.) $c^4 > a^4 + b^4$, (ii.) $a^4 + b^4 > c^4 > a^4$, (iii.) $c^4 = a^4$,
 (iv.) $a^4 > c^4 > b^4$, (v.) $c^4 = b^4$, (vi.) $b^4 > c^4$.

- (6) Draw the curves

- (i.) $\tan^{-1}x/a + \tan^{-1}y/b = \tan^{-1}c$;
 (ii.) $\tanh^{-1}x/a + \tanh^{-1}y/b = \tanh^{-1}c$;
 (iii.) $(\tan x)^2 + (\tan y)^2 = \frac{1}{3}$, or 3.

153. *Polar Coordinates.*

In discussing properties of a curve connected with straight lines radiating from any origin O , it is convenient to change to polar coordinates (r, θ) by putting (§ 22)

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Substituting in the rational integral equation (A), p. 315,

$$u_0 r^{-n} + (A \cos \theta + B \sin \theta) r^{-n+1}$$

$$+ (a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta) r^{-n+2} + \dots = 0;$$

an equation of the n^{th} degree in r^{-1} ; so that a straight line cuts a curve of the n^{th} degree in n points, some pairs of which however may be imaginary.

Denoting by $r_1, r_2, r_3, \dots, r_n$ the roots in r of this equation and their *harmonic mean* by r , then

$$r^{-1} = (r_1^{-1} + r_2^{-1} + \dots + r_n^{-1})/n = -(A \cos \theta + B \sin \theta)/nu_0,$$

$$\text{or} \quad Ar \cos \theta + Br \sin \theta + nu_0 = 0,$$

$$Ax + By + nu_0 = 0, \text{ or } u_1 + nu_0 = 0,$$

the equation of a straight line, the locus of a point P , such that OP is the harmonic mean of OP_1, OP_2, \dots, OP_n , where P_1, P_2, \dots, P_n are the n points in which the straight line cuts the curve; this straight line is called the *polar line* of O , by analogy with the polar line of a conic section, with which it coincides when $n=2$.

The straight line $u_1 + u_0 = 0$ is the locus of P when

$$r^{-1} = r_1^{-1} + r_2^{-1} + \dots + r_n^{-1}.$$

In the same way the *polar conic* of O ,

$$u_2 + (n-1)u_1 + \frac{n(n-1)}{1 \cdot 2}u_0 = 0,$$

is the locus of P when

$$\Sigma \left(\frac{1}{OP} - \frac{1}{OP_r} \right) \left(\frac{1}{OP} - \frac{1}{OP_s} \right) = 0;$$

and so on.

Writing, as usual, u for $1/r$, then since for the points of intersection with the curve,

$$\Sigma u = -(A \cos \theta + B \sin \theta)/nu_0,$$

therefore $\Sigma(d^2u/d\theta^2 + u) = 0$.

But $d^2u/d\theta^2 + u = c \operatorname{cosec}^3 \phi$,

by § 93, where c denotes the curvature, and ϕ the radial angle at which the vector OP cuts the curve; so that

$$\Sigma c \operatorname{cosec}^3 \phi = 0,$$

at the points of intersection of OP with the curve. (Dr. Routh, *Quarterly J. of Math.*, xxiv., p. 257.)

We have found that in a central field of force (§ 84), the acceleration to the centre

$$P = h^2 u^2 (d^2u/d\theta^2 + u),$$

so that the orbit is a straight line, and $d^2u/d\theta^2 + u = 0$, when $P = 0$; but the orbit is *concave* to the origin when P and therefore $d^2u/d\theta^2 + u$ is positive; and *convex* when they are negative; and at a *point of inflexion*, where the curve changes from *concavity* to *convexity*, or *vice versa*, $d^2u/d\theta^2 + u$ vanishes and changes sign.

Definition.—A curve is said to be *concave* with respect to a point or line when it lies on the same side of its tangent as the point or line in the neighbourhood of the point of contact; and *convex* when it lies on the opposite side of its tangent; and at a *point of inflexion* the curve crosses its tangent.

In interpreting the above, some of the ρ 's must be negative, and we shall take ρ as positive or negative according as the curve is concave or convex to the origin.

Incidentally we deduce that the three points of inflexion on a cubic lie in a straight line.

154. *Equation of the Chord, Tangent, Asymptote, and Normal of a Curve in Polar Coordinates.*

It is convenient to employ u , the reciprocal of r (§ 23), and now the equation of a straight line can be written

$$u = A \cos \theta + B \sin \theta ;$$

or, more generally,

$$u = A \cos (\theta - \alpha) + B \sin (\theta - \alpha),$$

equivalent in Cartesian coordinates to

$$1 = A(x \cos \alpha + y \sin \alpha) + B(y \cos \alpha - x \sin \alpha),$$

an equation of the first degree in x and y , and therefore the equation of a straight line.

To find the equation of the chord of the curve, whose equation is $u = f\theta$, which passes through the two points whose vectorial angles are $\alpha \pm \beta$, we must determine A and B from the equations

$f(\alpha + \beta) = A \cos \beta + B \sin \beta$, $f(\alpha - \beta) = A \cos \beta - B \sin \beta$;
and therefore

$$A = \frac{f(\alpha + \beta) + f(\alpha - \beta)}{2 \cos \beta}, \quad B = \frac{f(\alpha + \beta) - f(\alpha - \beta)}{2 \sin \beta};$$

so that the equation of the chord is

$$u = \frac{f(\alpha + \beta) + f(\alpha - \beta)}{2 \cos \beta} \cos (\theta - \alpha) + \frac{f(\alpha + \beta) - f(\alpha - \beta)}{2 \sin \beta} \sin (\theta - \alpha) \dots (i.)$$

To determine the tangent where $\theta = \alpha$, put $\beta = 0$; then

$$\text{lt} \frac{f(\alpha + \beta) + f(\alpha - \beta)}{2 \cos \beta} = f\alpha, \quad \text{lt} \frac{f(\alpha + \beta) - f(\alpha - \beta)}{2 \sin \beta} = f'\alpha;$$

and therefore the equation of the tangent at $\theta = \alpha$ is

$$u = f\alpha \cdot \cos (\theta - \alpha) + f'\alpha \cdot \sin (\theta - \alpha) \dots \dots \dots (ii.);$$

so that, if $OP = 1/f\alpha$, then $Ot = -1/f'\alpha$ (fig. 10);

and looking along OP from O , then Ot must be drawn to the right if $f'\alpha$ is negative; to the left if positive.

Suppose $f\alpha = 0$, but $f'\alpha$ is finite; the point of contact is then at an infinite distance from O , but the tangent remains at a finite distance, and the tangent is therefore

an asymptote, its equation being

$$u = f'a \cdot \sin(\theta - \alpha) \dots \dots \dots (\text{iii.}).$$

If $f'a = \infty$, the asymptote passes through O , and its equation is given by $\theta = \alpha$.

The equation of the normal will be

$$u = fa \cdot \cos(\theta - \alpha) - \frac{(fa)^2}{f'a} \sin(\theta - \alpha) \dots \dots \dots (\text{iv.});$$

for it is the equation of a straight line through the point of contact of the tangent, where $\theta = \alpha$; and this straight line is perpendicular to the tangent, as is readily seen by changing to Cartesians.

When the equation of the curve is given in the form $r = F\theta$, the equation of the normal will be found to be

$$\frac{1}{r} = \frac{\cos(\theta - \alpha)}{F\alpha} + \frac{\sin(\theta - \alpha)}{F'\alpha} \dots \dots \dots (\text{v.});$$

so that $Og = F'\alpha$, if $OP = F\alpha$ (fig. 10).

For instance, the equation of the chord, tangent, and normal, of the conic $lu = 1 + e \cos \theta$, are

$$lu = e \cos \theta + \sec \beta \cos(\theta - \alpha), \quad lu = e \cos \theta + \cos(\theta - \alpha).$$

$$lu = \frac{1 + e \cos \alpha}{\sin \alpha} \{e \sin \theta + \sin(\theta - \alpha)\}.$$

Examples.

(1) Find the asymptotes of

(i.) $r = a \sec \theta, b \operatorname{cosec} \theta, a \sec \theta + b \operatorname{cosec} \theta, a \sec(\theta - \alpha),$
 $b \operatorname{cosec}(\theta - \beta), a \sec(\theta - \alpha) + b \operatorname{cosec}(\theta - \beta).$

(ii.) $r = a \tan \theta, a \cot \theta.$

(iii.) $r = a(\sec \theta - \cos \theta), a(\sec \theta + \tan \theta);$ and determine their equations in x and y .

(iv.) $r = \sec 2\theta, \operatorname{cosec} 2\theta, \sec 3\theta, \dots \sec n\theta, \operatorname{cosec} n\theta.$

(v.) $r^2 = a^2 \sec 2\theta, a^2 \operatorname{cosec} 2\theta.$

(vi.) $r = a + b \operatorname{cosec} \theta$ (the *conchoid* of Nicomedes).

(vii.) $r = a/\theta, a\theta/(\theta - \alpha), a\theta^2/(\theta^2 - \alpha^2), a \sin m\theta/\sin \theta.$

(viii.) $l/r = \pm 1 + \sec \beta \cos \theta$ (the hyperbola).

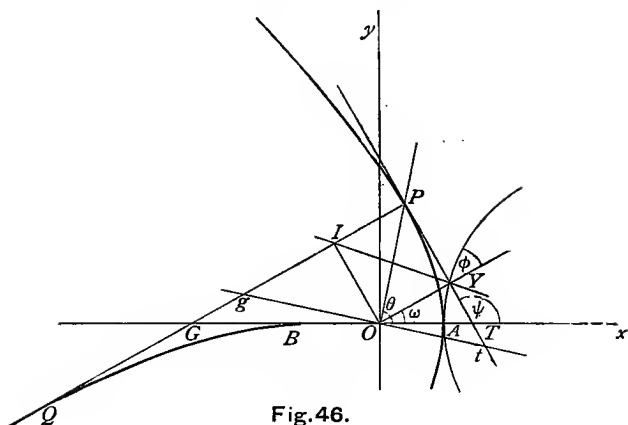


Fig. 46.

155. Pedal Curves.

The locus of Y , the foot of the perpendicular on the tangent of a curve drawn from the origin O , is called the *pedal* of the curve with respect to O , and O is called the *pole* of the pedal.

Thus with respect to a focus, the pedal of an ellipse or hyperbola is the auxiliary circle, and the pedal of a parabola is the tangent at the vertex of the parabola.

If OYP is a rigid right angle, of which OY passes through O , and YP touches the curve, then I , the foot of the perpendicular from O on the normal at P , is the *centre of instantaneous rotation* of the right angle (§ 21); so that IY is the normal of the locus of Y .

Since IY is a diameter of the circle described on OP as diameter, it follows that the envelope of circles described on the variable vector OP as diameter is the pedal with respect to O , the locus of Y .

This can easily be generalized for the case of a rigid angle PYP' touching two fixed curves, at P and P' ; the centre of instantaneous rotation will be at I , the point of intersection of the normals at P and P' , and therefore IY is the normal of the locus of Y ; and since the circle circumscribing the triangle PYP' has the same normal at Y , it follows that the locus of Y is the envelope of these circles, described round the triangle PYP' .

Returning to the pedal of the curve (fig. 46), then since the angle OYI is equal to the angle OPI in the same segment, it follows that the pedal curve cuts OY at the same angle as the curve cuts OP , or the pedal and the curve have the same radial angle ϕ (§ 22) at corresponding points.

Therefore, denoting OY by p , and the angle AOY by ω ,

$$\cot \phi = \frac{YP}{OY} = \frac{dp}{pd\omega}, \text{ so that } YP = \frac{dp}{d\omega}.$$

We denote OI by q , so that the polar coordinates of I are (q, ψ) and $q = dp/d\omega$, $\psi = \omega + \frac{1}{2}\pi$; and since Q , the centre of curvature at P , is the point of contact of the normal PI of the curve AP , or of the tangent PI of the evolute BQ at Q , therefore the locus of I is the pedal of the evolute with respect to O , and $IQ = dq/d\psi = d^2p/d\omega^2$.

Therefore the radius of curvature at P ,

$$ds/d\omega = \rho = p + dq/d\psi = p + d^2p/d\omega^2 = p + qdq/dp.$$

The equation of the tangent YP being

$$x \cos \omega + y \sin \omega = p;$$

and of the normal PI being

$$x \cos \psi + y \sin \psi = q \text{ or } -x \sin \omega + y \cos \omega = dp/d\omega;$$

it is therefore derived from the tangent by differentiating with respect to ω .

The equation of the normal to the evolute at Q will therefore be $-x \cos \omega - y \sin \omega = d^2p/d\omega^2$;
so that the coordinates of Q are

$$-\sin \omega \frac{dp}{d\omega} - \cos \omega \frac{d^2p}{d\omega^2}, \cos \omega \frac{dp}{d\omega} - \sin \omega \frac{d^2p}{d\omega^2}.$$

The relation $p=f\omega$, connecting p and ω for the curve AP , is called its *tangential polar* equation; and $p=f\omega$ is the polar equation of the pedal curve AY , with p and ω as polar coordinates; and then $q=f'(\omega+\frac{1}{2}\pi)$ will be the polar equation of the pedal of the evolute BQ .

The original curve AP is the *envelope* (§ 105) of

$$x \cos \omega + y \sin \omega = f\omega;$$

and with reference to the curve AY , the curve AP is called the *first negative pedal* of AY with respect to O . Thus the first negative pedal of a circle is an ellipse or hyperbola, according as the pole is inside or outside the circle; the first negative pedal of a straight line is a parabola.

For instance the directions of motion of the parts of a rope in contact with a moving pulley are at any instant tangents to a conic, of which a focus is at the instantaneous centre of rotation of the pulley.

We may take the pedal of the pedal, positive or negative, with respect to the pole O , any number of times n , and then we obtain a curve called the n^{th} pedal, positive or negative.

Suppose rays of light proceeding from O are incident on a reflecting curve AP ; the reflected ray will pass through Z on OY , produced so that $OZ=2OY$, and will be parallel to YI , and therefore normal to the locus of Z ; the envelope of the reflected rays, called the *katakaustic* (§ 105), is therefore the evolute of the locus of Z , a curve double the size of the pedal curve AY .

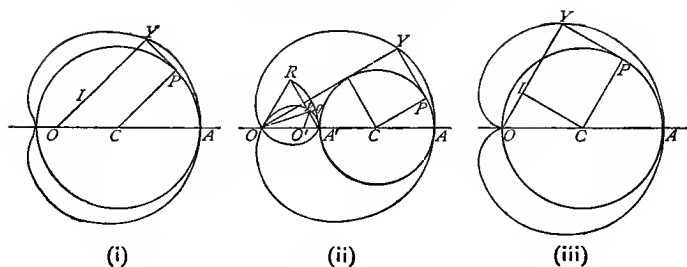


Fig. 47

156. *Limaçons, the Pedals of a Circle.*

Consider the pedal of a circle, centre C and radius a , with respect to any point O , where $OC = b$.

Then $p = OY = IY + OI = a + b \cos \omega$,

the equation connecting the polar coordinates p and ω of the locus of Y , and this curve is called a *limaçon*.

The limaçon is called a *conchoid* of a circle, because it is described by producing the vector of the circle $r = b \cos \theta$ a constant distance a .

The equation of the limaçon may be written

$$p = \pm a + b \cos \omega,$$

corresponding to parallel tangents of the circle; so that the chord of the limaçon through O is of constant length $2a$.

If $b < a$, O is inside the circle, and the pedal consists of a single oval curve (fig. 47, i.). This oval has points of inflexion, if $b > \frac{1}{2}a$.

If $b > a$, O is outside the circle, and the pedal is looped, having a double point O (fig. 47, ii.).

If $b = a$, O is on the circumference, and $p = a(1 + \cos \omega)$, the equation of a *cardioid* (fig. 47, iii.).

If in fig. 8 we fix the bar PQR , and move the cross OQR , then any point fixed in OQ or OR will describe a limaçon.

157. *Orthoptic and Isoptic Curves.*

If two tangents PR , QR to the curve APQ intersect at a constant angle α , in the point R , the locus of R is called an *isoptic* curve of the curve APQ ; and if the angle α is a right angle, the locus of R is called the *orthoptic* curve.

Thus the isoptic curve of a circle is a concentric circle the orthoptic locus of an ellipse or hyperbola is a circle sometimes called the *director* circle; the orthoptic locus of a parabola is its directrix, while the isoptic locus of a parabola is a confocal hyperbola.

The equation of the *isoptic* locus for tangents inclined at an angle α is obtained by eliminating ω between

$$x \cos(\omega + \frac{1}{2}\alpha) + y \sin(\omega + \frac{1}{2}\alpha) = f(\omega + \frac{1}{2}\alpha),$$

$$x \cos(\omega - \frac{1}{2}\alpha) + y \sin(\omega - \frac{1}{2}\alpha) = f(\omega - \frac{1}{2}\alpha);$$

and $\alpha = \frac{1}{2}\pi$ gives the *orthoptic* locus.

The normal RI of the isoptic locus at R will pass through I the point of intersection of the normals at the points P , Q ; since I is the centre of instantaneous rotation of the constant angle PRQ .

Examples.

- (1) Prove that the equation of the pedal of an ellipse with respect to the centre is

$$p^2 = a^2 \cos^2 \omega + b^2 \sin^2 \omega, \text{ or } (x^2 + y^2)^2 = a^2 x^2 + b^2 y^2.$$

Show that the pedal has points of inflexion when $b^2/a^2 < \frac{1}{2}$, or $e^2 > \frac{1}{2}$; and that, when $b=0$, the pedal reduces to two circles.

- (2) Prove that the pedal of a parabola with respect to the vertex is the *cissoid*

$$p = a(\sec \omega - \cos \omega), \text{ or } y^2 = x^3/(a-x).$$

- (3) Prove that the isoptic locus of a parabola is a hyperbola, of excentricity $\sec \alpha$; and explain how the two branches of the hyperbola are formed.

- (4) The isoptic locus of an ellipse is given by

$$(x^2 + y^2 - a^2 - b^2)^2 = 4 \cot^2 \alpha (a^2 y^2 b^2 + x^2 - a^2 b^2).$$
- (5) The isoptic locus of a cycloid is a trochoid.
- (6) The orthoptic locus of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $\frac{1}{2} \sqrt{2} a \cos 2\theta$.
- (7) The orthoptic locus of the cardioid is composed of a circle and a limaçon (Wolstenholme).
- (8) Prove that if $\rho_1, \rho_2, \rho_3, \dots$ are the radii of curvature of the envelopes of the sides of a polygon, whose sides are a, b, c, \dots then $a\rho_1 + b\rho_2 + c\rho_3 + \dots$ is equal to twice the area of the polygon.
- (9) Prove that the polar equation of the curve OP of § 119, in the neighbourhood of O , is

$$r = 2\rho\theta + \frac{4}{3}\rho\rho'\theta^2 + \dots$$
- (10) Prove that the pedal of an involute of a circle, with respect to the centre of the circle, is a spiral of Archimedes (fig. 36).

Apply this to the theory of a weighing machine, showing the weight on a dial provided with equal graduations, when the body is weighed against a fixed weight, suspended by a rope which wraps on the involute of a circle.

158. *Roulettes.*

When a curve, carrying a point P fixed to it, rolls on a straight line (or any given curve), the path traced out by the point P is called the roulette of P with respect to the straight line (or given curve).

Thus, when a circle rolls on a straight line, the roulette of a point on the circumference is a *cycloid*, and the roulette of any other point fixed in the plane of the circle is a *trochoid* (§ 21).

An *involute* of a curve (§ 95) is thus the roulette of a point on a straight line which rolls on the curve.

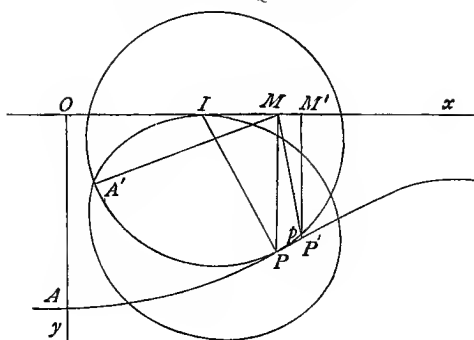


Fig. 48

A remarkable analogy, pointed out by Steiner, exists between the roulette of a point with respect to a straight line and the pedal of the rolling curve with respect to the point as pole. Steiner's Theorems assert that

(i.) The length of the arc of the roulette is equal to the length of the corresponding arc of the pedal ;

(ii.) The area bounded by an arc of the roulette, the ordinates at the ends of the arc, and the straight line on which the curve rolls is twice the area bounded by the corresponding arc of the pedal and the vectors from the origin to the ends of the arc.

For, if AP is the roulette of the point P when the curve is rolled on the straight line Ox (fig. 48), and if PM is the perpendicular from P on Ox , the tangent at I to the rolling curve, then relatively to P the locus of M is the pedal of the rolling curve with respect to P ; and therefore relatively to M the locus of P is the same curve, and the subnormal IM of the roulette is the q or $dp/d\omega$ of the rolling curve, or the polar subnormal of its pedal.

We may suppose the pedal $A'P$ rolled on the roulette AP , so that M is always vertically over P if Ox is horizontal; and the pedal, if loaded so that the centre of gravity is at M , will rest in neutral equilibrium on the roulette, provided the friction is sufficient, or else that teeth are cut, to prevent slipping.

The arc AP of the roulette will then be equal to the corresponding arc $A'P$ of the pedal, which is Steiner's first theorem.

The arc s in the roulette is therefore the same function of y as in the pedal of p .

Also if the pedal is rolled into a consecutive position so that M comes to M' , and the point p of the pedal comes into contact with the point P' of the roulette, then the element $MM'P'P$, which is the increment of area of the roulette, is ultimately double the element MPp , which is the increment of area of the pedal, or

$$\text{lt}(\text{area } MM'P'P)/\text{area } MPp = 2;$$

and therefore, by integration, the area $OMPA$ of the roulette is double the area $A'MP$ of the pedal, which is Steiner's second theorem.

Examples.

- (1) Draw the figures and compare the arcs and areas of the following pairs of curves, the first curve of a pair being fixed, and the second rolling on it, so that its pole describes a straight line Ox (or Oy); thus illustrating Steiner's Theorems.

- (i.) The parabola $y^2 = 2lx$, and the spiral $r = l\theta$.

- (ii.) The circle $x^2 + y^2 = a^2$, and the circle $r = a \cos \theta$.

(This principle is employed in the parallel motion of Deleuil's air pump. Deschanel, *Physics*.)

- (iii.) The ellipse $(x/a)^2 + (y/b)^2 = 1$,
and $r = b \cos(b\theta/a)$ or $a \cos(a\theta/b)$.
- (iv.) The hyperbola $(x/a)^2 - (y/b)^2 = 1$,
and $r = a \cosh(a\theta/b)$ or $b \sinh(b\theta/a)$.
- (v.) The exponential curve $y = be^{x/c}$,
and the reciprocal spiral $r = c/\theta$.
- (vi.) The cycloid and the cardioid.
- (vii.) The trochoid and the limaçon.
- (viii.) The straight line $y = x \tan \alpha$,
and the equiangular spiral $r = ae^{\theta \tan \alpha}$.
- (ix.) The catenary $y = a \cosh x/a$,
and the straight line $r = a \sec \theta$.
- (x.) The modified catenary $y = b \cosh x/a$,
and the Cotes's spiral $r \cos(b\theta/a) = b$.
- (xi.) The sinusoid $y = b \cos x/a$,
and the Cotes's spiral $r \cosh(b\theta/a) = b$.
- (xii.) The sinusoid $y = a - ae \cos(x/b)$,
and the ellipse $l/r = 1 + e \cos \theta$.
- (xiii.) The hyperbola $xy = c^2$, and the spiral $r^2\theta = \frac{1}{2}c^2$.
- (xiv.) The semicubical parabola $ay^2 = x^3$,
and the spiral $r\theta^3 = 8a$.
- (xv.) The curves $(x/a)^m = (y/b)^n$ and $r^{m-n}\theta^m = c^{m-n}$.
- (xvi.) The curves $x^m y^n = a^{m+n}$, and $r^{m+n}\theta^m = c^{m+n}$.

Rectify these curves, when possible; and prove that s is the same function of y in the first curve as of r in the second; also that ydy/dx and ydx/dy , the subnormal and subtangent, are the same functions of y , as $dr/d\theta$ or $r^2 d\theta/dr$, the polar subnormal and subtangent are functions of r .

(2) Prove that the roulette of the pole of

$l/r = 1 + \sec a \cos(\theta \sin a)$, or $1 + \operatorname{sech} a \cosh(\theta \sinh a)$,
with respect to a straight line, is a circle.

- (3) Prove that the curvature of the roulette with respect to a straight line is $d \sin \phi / dp$, where ϕ is the radial angle and p the perpendicular on the tangent from the pole on the rolling curve.
- (4) Prove that if the centre of an ellipse is fixed at a distance b from a plane, and the ellipse is rolled on the plane, the point of contact will describe the Cotes's spiral $r \cosh (ae\theta/b) = ae$.

159. Centroides.

When a moving plane figure slides or turns on another plane, which may be considered fixed, then a point I in the moving plane can always be found which has no velocity; this point I is called the *centre of instantaneous rotation* (§ 21), and the relative motion of the two planes is assigned by the position of I and by the angular velocity n of the moving plane round I .

The point I will in the general case describe a curve in the moving plane and a curve in the fixed plane, and the motion of the moving plane will be given by rolling the first curve on the second; so that any point carried by the rolling plane will describe a roulette of the first curve with respect to the second curve; these curves described by I are now called the *centroides* of the relative motion of the two planes.

Another point J can always be found of which the *acceleration* is zero; and now the acceleration of any other carried point P at a distance r from J will be composed of component accelerations $n^2 r$ towards J and $\dot{n} r$ perpendicular to JP , \dot{n} denoting the angular acceleration dn/dt .

The points whose trajectories have zero curvature, and which are therefore at this instant describing straight paths (as M in fig. 48), or rather are passing through points of inflexion on their trajectories, are obtained by putting $\theta = \alpha$, and therefore lie on a circle IQJ passing through I and J , called the *circle of inflexions* (the circle PMI in fig. 48); and this circle will touch the centrodes at I , since the acceleration at M is in the normal.

Now if IP meets this circle in Q , then the curvature of the trajectory of P is

$$\begin{aligned} 1/R &= n^2 r \sin(\alpha - \theta) / v^2 \sin \alpha \\ &= n^2 \cdot PQ / v^2 = PQ / PI^2. \end{aligned}$$

Thus

$$PP' = PI^2 / PQ,$$

$$IP' = PI^2 / PQ - PI = PI \cdot IQ / PQ,$$

or

$$\frac{1}{IP'} = \frac{PQ}{PI \cdot IQ} = \frac{1}{IQ} - \frac{1}{PI},$$

or

$$\frac{1}{PI} + \frac{1}{IP'} = \frac{1}{IQ} = \frac{\sec \phi}{ID},$$

where ID is the diameter of the circle of inflexions, and where ϕ now denotes the angle between IP and the common normal of the centrodes at I .

A point P inside the circle of inflexions will thus describe a trajectory *convex* with respect to I , but *concave* if the point is outside this circle; so that the rolling centrode, if loaded so that its centre of gravity is at P , will be in stable or unstable equilibrium with IP vertical according as P is inside or outside this circle.

The diameter ID of the circle of inflexions is inferred by placing the carried point P for a moment at C , the centre of curvature of the rolling centrode at I , when it is easily seen that the centre of curvature of the roulette

through C will be at C' , the corresponding centre of curvature at I of the fixed centrode; and now, with $\phi=0$,

$$\frac{1}{ID} = \frac{1}{CI} + \frac{1}{IC'} = \frac{1}{\rho} + \frac{1}{\rho'},$$

or $ID = \rho\rho' / (\rho + \rho')$, where ρ, ρ' denote the radii of curvature CI, IC' of the rolling and fixed centrodes, reckoned positive when the centrodes are convex to each other.

$$\text{Now} \quad \frac{1}{PI} + \frac{1}{IP'} = \sec \phi \left(\frac{1}{\rho} + \frac{1}{\rho'} \right),$$

and the symmetry of this relation shows that P is the centre of curvature of the roulette of P' with respect to the former moving centrode; also that $CP, C'P'$ intersect in a point T on IT the perpendicular to IP .

When a curve rolls symmetrically on an equal curve, the roulette of any point will be similar to the pedal of the curve with respect to P , but enlarged to twice the scale; the reason being that the reflexion of the carried point P in the tangent at I is a fixed point.

Suppose for instance, as drawn in fig. 49, that the fixed and rolling centrodes are equal ellipses, and that the carried point P is at one of the foci; the roulette of P will be a circle, and the centre of curvature P' will lie at a focus of the fixed ellipse.

We can pivot these ellipses at the other foci O and O' , and now revolve them in contact with each other; teeth may be cut to prevent slipping, and the revolving foci P and P' may be connected by a link to prevent separation. This mechanism is sometimes employed, and the ellipses roll on each other as if connected with a crossed parallelogram of bars $OP, PP', P'O', O'O$.

160. *The Area of a Roulette.*

As I moves along the rolling and fixed centrodes with the same velocity ds/dt , the moving plane will turn in the time dt through an angle $d\omega$, which is the sum of the curvatures ds/ρ and ds/ρ' of the equal arcs ds of the rolling and fixed centrodes; so that

$$n = \frac{d\omega}{dt} = \left(\frac{1}{\rho} + \frac{1}{\rho'} \right) \frac{ds}{dt}.$$

The normal PI of the roulette of P now sweeps out an area, denoted by (PI) , at a rate which, by Guldin's Theorem generalized (§ 62), will be measured by the product of PI and of the component velocity of the middle point of PI perpendicular to PI .

This component velocity, being the arithmetic mean of the velocities of I and P in the same direction, is equal to

$$\frac{1}{2} \cos \phi (ds/dt) + \frac{1}{2} n \cdot PI;$$

so that

$$\frac{d(PI)}{dt} = \frac{1}{2} PI \cos \phi \frac{ds}{dt} + \frac{1}{2} n \cdot PI^2 = \frac{1}{2} p \frac{ds}{dt} + \frac{1}{2} \left(\frac{1}{\rho} + \frac{1}{\rho'} \right) PI^2 \frac{ds}{dt}.$$

Of these two terms, the first is the rate at which PI sweeps out polar area on the rolling centrode with respect to the pole P (§ 56); and the second represents the growth of the M.I. round P of the perimeter of the rolling centrode, supposing the perimeter replaced by a wire of variable density $\frac{1}{2}(c+c')$, the arithmetic mean of the curvatures $c=1/\rho$ and $c'=1/\rho'$ of the rolling and fixed centrodes.

Thus if the rolling centrode returns to its original position after one or more complete revolutions, the area of the roulette of P exceeds the sum of the areas of the rolling and fixed centrodes by the moment of inertia of this perimeter of variable density of the rolling centrode.

By taking the carried point P at G , the centre of gravity of this perimeter, we obtain the minimum moment of inertia (§ 64), and therefore the roulette of minimum area; and the area (P) of the roulette of P will exceed the area (G) of the roulette of G by $M \cdot GP^2$, where M is the mass distributed on the perimeter.

But $d\omega/dt$ denoting the angular velocity of the rolling centrode,

$$M = \int \frac{1}{2}(c + c')ds = \int \frac{1}{2}d\omega = n\pi,$$

for n complete revolutions of the rolling centrode; so that

$$(P) - (G) = n\pi \cdot GP^2.$$

161. *Theorems of Holditch, Elliott, Leudesdorf, and Kempe.*

A simple relation connects the areas (P) , (A) , (R) swept out by three points P , A , R in a straight line on a bar, which we may take to be the bar of the planimeter of fig. 45, carried round by the rolling centrode, and making n complete revolutions.

For if GK is the perpendicular from G on AP produced, then

$$(P) - (A) = n\pi(GP^2 - GA^2) = n\pi(KP^2 - KA^2)$$

$$= n\pi b(KP + KA),$$

and $(A) - (R) = n\pi c(KA + KR),$

since $AP = b$, $RA = c$; so that

$$\frac{(A) - (R)}{c} - \frac{(P) - (A)}{b} = n\pi(KR - KP) = n\pi \cdot PR,$$

or $(b + c)(A) - b(R) - c(P) = n\pi bc(b + c);$

this is called *Holditch's Theorem*.

If the closed contours described by A and P lie entirely outside each other, the bar AP can only oscillate between two extreme positions, and $n = 0$.

In using the planimeter, the pivot O is generally fixed outside the contour described by P , and the joint A oscillates on the arc of a circle of radius a , so that

$$(A)=0 \text{ and } n=0; \text{ and thus } b(R)+c(P)=0,$$

attending to the sign of the area.

Thus if P describes a circle of radius r not enclosing O , the motion of the bars OA , AP is similar to that of the beam and connecting rod of a steam engine; and now $(R)=-\pi r^2 c/b$; this is independent of the length of the beam, and is therefore the same for the ordinary direct action steam engine, as applied in the locomotive engine, where A oscillates in a straight line.

A new form of Planimeter has been lately brought out, in which the end A of the bar AP is constrained to move in a straight or curved slot, while the pencil P is carried round the area to be measured; but now the area is registered by the motion of a wheel W which can slide and turn on a graduated round bar CD , projecting at right angles to AP from a point C , which we may take to be the middle point of AP .

The rate at which AP sweeps out area is the product of AP into the component velocity of C perpendicular to AP ; and this component velocity is equal to the velocity with which the wheel W slides along CD ; the wheel being supposed to roll, but not to slide on the paper; in this manner the sliding motion of the roller R on the paper of Amsler's Planimeter is obviated.

Mr. Elliott has generalized Holditch's theorem by showing that the bar PAR may be replaced by an elastic thread, which stretches, but keeps the ratio b/c constant, when a similar Theorem holds.

Mr. Leudesdorf has extended the theorem to the relation connecting the areas of the roulettes (A) , (B) , (C) of any three points A , B , C with the area (P) of the roulette of any fourth point P ; and he finds that

$(P) - x(A) - y(B) - z(C) = n\pi$ times the rectangle on the segments of a chord through P of the circle circumscribing the triangle ABC , when x , y , z denote the ratios of the triangles PBC , PCA , PAB to the triangle ABC .

For $(P) - (A) = n\pi(GP^2 - GA^2), \dots$;
 so that $(P) - x(A) - y(B) - z(C)$
 $= n\pi(GP^2 - x \cdot GA^2 - y \cdot GB^2 - z \cdot GC^2)$
 $= n\pi(x \cdot BC^2 + y \cdot CA^2 + z \cdot AB^2)$

which can be shown to lead to Mr. Leudesdorf's result; and reduces to Holditch's Theorem when P lies in a side BC .

Mr. Kempe points out that the locus of P , for which (P) is zero, is a circle; and the locus of P , for which (P) is constant, is a concentric circle, exceeding in area the first circle by an amount proportional to (P) ; but that if $n=0$, this system of concentric circles must be replaced by a system of parallel straight lines.

(*Messenger of Mathematics*, vol. vii., 1878.)

162. *The Envelope of a Carried Line.*

Similar theorems hold for the envelope of a carried line Nx , fixed in the moving centrode and carried round by it.

The point of contact N on the envelope is the foot of the perpendicular IN drawn from I on the carried line; and now if we denote by y the ordinate IN of the point I of the rolling centrode with respect to the line Nx , by x the corresponding abscissa on the line Nx , and by S the length of the arc described by N ; then

$$\frac{dS}{dt} = \frac{dx}{dt} + y \frac{d\omega}{dt},$$

so that $S - x = \int y d\omega = \int \frac{1}{2}(c + c') y ds$;

or S the length of the arc of the envelope exceeds x the projection of the corresponding arc of the rolling centre on the carried line by the moment about the carried line of the same distribution of density $\frac{1}{2}(c + c')$ as before.

Carried lines which have envelopes of the same length will therefore touch a circle, with the centre of gravity G of the perimeter as centre.

Again, if ϕ' denotes the angle between IN and the common normal of the centrodes at I , the radius of curvature R' of the envelope of the carried line at N is given by

$$\begin{aligned} R' &= \frac{dS}{d\omega} = \frac{dx}{d\omega} + y = \cos \phi' \frac{ds}{d\omega} + y \\ &= Q'I + IN = Q'N, \end{aligned}$$

supposing NI meets in Q' the circle $IQ'D'$, which is the reflexion of the circle of inflexions in the common tangent of the centrodes at I .

Therefore, for all carried lines passing through D' , $R' = 0$, and the envelope has a cusp lying on this circle, called for this reason the *circle of cusps*; while in addition the circle is the locus of the centres of curvature of all carried lines, the centre of curvature Q' being obtained by dropping the perpendicular $D'Q'$ on IN .

The normal NI of the envelope will now sweep out area (NI) at a rate

$$\frac{d(NI)}{dt} = \frac{1}{2}y \left(\frac{dx}{dt} + \frac{dS}{dt} \right) = y \frac{dx}{dt} + \frac{1}{2}y^2 \frac{d\omega}{dt};$$

so that the area swept out by the normal NI exceeds the area swept out on the rolling centre by the ordinate NI by the M.I. of the perimeter, of variable density $\frac{1}{2}(c + c')$, round the carried line.

Thus from Theorem II., p. 128, it follows that if two carried lines intersect at right angles in P , the sum of the areas of their envelopes exceeds the area of the roulette of P by the area of the rolling centrode.

Carried lines which sweep out equal areas are such that the M.I. of the perimeter round them is the same; they are therefore *equimomental* lines, and therefore tangents of an *equimomental ellipse*; and by varying the area we obtain a system of equimomental confocal ellipses, by well known theorems.

(Besant, *Roulettes and Glissettes*;

Kempe, *Messenger of Mathematics*; *Nature*, 1878;

Darboux, *Bulletin des Sciences Mathématiques*, 1878;

Larmor, *Nature*, 1881; *Proc. Cam. Phil. Soc.*, 1890;

Minchin, *Uniplanar Kinematics*, 1882.)

163. When the fixed centrode becomes a straight line, $c' = 0$; and the area swept out by the normal PI (fig. 48) in a complete revolution of the rolling curve is the same as that swept out by the ordinates MP , and is therefore double the area of the pedal of the rolling curve with respect to P .

Now, if the rolling curve is coated with matter of linear density $\frac{1}{2}c$ per unit of length, the mass will be

$$\int \frac{1}{2}cds = \int_0^{2\pi} \frac{1}{2}d\omega = \pi;$$

and therefore the area of the roulette of P will exceed the area of the roulette of G , the centre of gravity of this wire, by $\pi \cdot GP^2$; and consequently the area of the pedal of the rolling curve with respect to P will exceed the area of the pedal with respect to G by $\frac{1}{2}\pi \cdot GP^2$.

Thus, for instance, the pedal of a circle with respect to its centre being the circle itself, will have an area πa^2 ; and consequently the area of the limaçon $p = a + b \cos \omega$ will be $\pi a^2 + \frac{1}{2}\pi b^2$, while the area of the corresponding trochoid will be $2\pi a^2 + \pi b^2$; and with $a = b$, the area of the cardioid $p = a(1 + \cos \omega)$ will be $\frac{3}{2}\pi a^2$, and of the cycloid $3\pi a^2$.

In a roulette with respect to a straight line,

$$\frac{1}{R-N} + \frac{1}{N} = \frac{\sec \phi}{\rho},$$

or

$$NR - N^2 = \rho R \cos \phi,$$

or

$$\frac{1}{R} = \frac{1}{N} - \frac{\rho \cos \phi}{N^2}.$$

If the rolling curve is an ellipse and the carried point at a focus, then (fig. 48)

$$\rho \cos \phi = N(2a - N)/a = 2N - N^2/a,$$

so that

$$\frac{1}{R} + \frac{1}{N} = \frac{1}{a}.$$

If this roulette sweeps out a surface by revolution round the fixed straight line, the curvature of this surface will be everywhere $1/a$, a constant; a soap-bubble film of revolution will assume the shape of this surface, or a surface similarly generated by the roulette of the focus of a hyperbola or parabola.

For instance, the roulette of the focus of a parabola is a catenary and we obtain the catenoid of fig. 16.

Again, the roulette of the pole of the involute of a circle with respect to a straight line is a parabola; and the roulette of the centre of the rectangular hyperbola $r^2 \cos 2\theta = a^2$ is a curve in which $R = \frac{1}{2}N$, or $Ry = \frac{1}{2}a^2$, an *elastica* or *lintearia*.

164. *Epicycloids and Hypocycloids.*

These curves are the roulettes of a point P on the circumference of a circle which rolls on the outside or inside of a fixed circle.

Let O denote the centre and a the radius of the fixed circle, C the centre and c the radius of the rolling circle, and I the point of contact of the circles; then IP is the normal of the roulette of P , because I is the centre of instantaneous rotation of the rolling circle (fig. 50).

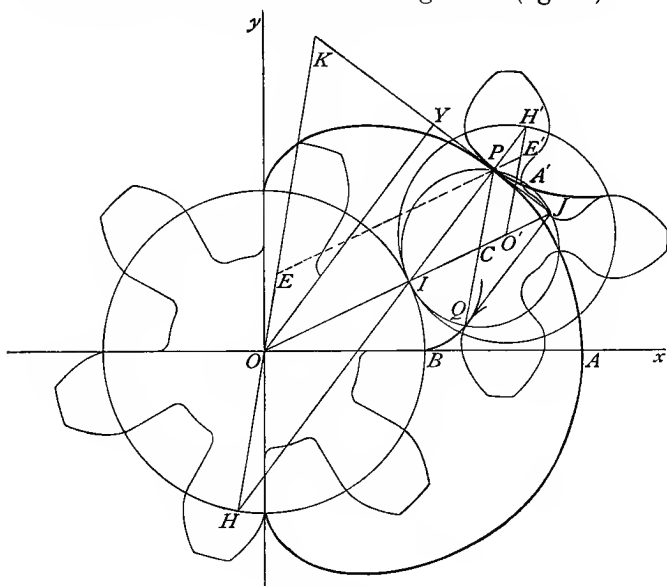


Fig. 50.

Draw the diameter PCQ of the rolling circle, and suppose Q originally in contact with the fixed circle at B , and that P is then at A ; A is then an apse and B a cusp of an epicycloid (§ 104).

Denoting the angle xOI by θ (radians), then the arc $IB = a\theta = \text{arc } IQ$, so that the angle $ICQ = a\theta/c$; and the coordinates of P in terms of θ , for the epicycloid, are

$$\begin{aligned}x &= (a+c)\cos \theta + c \cos(1+a/c)\theta, \\y &= (a+c)\sin \theta + c \sin(1+a/c)\theta;\end{aligned}$$

and for a hypocycloid, change c into $-c$.

Thence we find, for the epicycloid,

$$\frac{ds}{d\theta} = 2(a+c)\cos \frac{a\theta}{2c},$$

and integrating, the arc AP

$$= s = 4\frac{c}{a}(a+c) \sin \frac{a\theta}{2c};$$

while the $\text{arc } BQ = 4\frac{c}{a}(a+c) \text{vers} \frac{a\theta}{2c}$.

With the notation of p and ω of § 154,

$$p = (a+2c)\cos(a\theta/2c), \text{ and } \omega = (1+a/2c)\theta;$$

so that $p = (a+2c)\cos\{a\omega/(a+2c)\}$,

the polar equation of the pedal of an epicycloid, of the form $r = b \cos m\theta$.

$$\text{Also } PY = -\frac{dp}{d\omega} = a \sin \frac{a\omega}{a+2c} = a \sin \frac{a\theta}{2c};$$

so that $r^2 = OY^2 + PY^2$

$$= (a+2c)^2 \cos^2 \frac{a\theta}{2c} + a^2 \sin^2 \frac{a\theta}{2c}$$

$$\text{or } r^2 - a^2 = 4c(a+c) \cos^2 \frac{a\theta}{2c} = \frac{4c(a+c)}{(a+2c)^2} p^2,$$

of the form $r^2 - a^2 = (1-m^2)p^2$, $m = a/(a+2c)$,

the relation connecting p and r in the epicycloid; proving incidentally that the roulette of the centre of an epicycloid with respect to a straight line is an ellipse.

If $m=0$, then $c=\infty$, and the curve becomes the involute of a circle, in which $p^2 = r^2 - a^2$.

Again
$$s = \frac{4c}{a}(a+c) \sin \frac{a\omega}{a+2c},$$

of the form $s = l \sin m\psi$, with ψ for ω , l for $4c(a+c)/a$, and m for $a/(a+2c)$, as in § 104.

And
$$\rho = \frac{ds}{d\omega} = \frac{4c(a+c)}{a+2c} \cos \frac{a\omega}{a+2c} = \frac{rdr}{dp} = (1-m^2)p.$$

165. *The Teeth of Wheels.*

By cutting teeth on wheels we can make them engage and transmit power without slipping; we thus secure the condition called *perfect roughness* by the theoretical mathematician; perfect smoothness between two bodies on the other hand is sought practically by the interposition of wheels.

Suppose O' to be the fixed centre of a wheel of radius $O'I = a'$, which is to be made to revolve in contact with the wheel of centre O and radius a , without slipping.

If the circle, centre C and radius c , rolls inside this circle, centre O' and radius a' , and describes the hypocycloid $A'P$, then if the epi- and hypo-cycloids AP and $A'P$ start with the vertices A and A' in contact, the two curves will roll and slide on each other (fig. 50) so that the common normal at P passes through I , and therefore the constant velocity ratio of the wheels is maintained; for this reason the wheels of teeth are shaped by epi- and hypo-cycloids.

Only a small portion of each curve in the neighbourhood of a cusp is made use of to form a tooth; and the tooth is completed on the circle O' by a portion of an epicycloid, and on the circle O by a portion of a hypocycloid, each described by the rolling of a circle of the same radius c' .

For instance, if $c' = \frac{1}{2}a$, then the hypocycloid in the circle O is given by $x=0$, $y=a \sin \theta$; so that the hypocycloid degenerates into a straight line, a diameter; and the inside portion of the tooth is straight and radial.

Any number of change wheels of a lathe may be made to work together indiscriminately, provided the teeth are shaped by rolling circles of the same radius c .

In fig. 50 we have taken $a/a' = \frac{6}{4}$, and $c=c' = \frac{1}{2}a$; and four teeth on the circle O engage with six teeth on O' .

When we make c and c' infinite, the teeth are shaped by *involute*s of the circle O and O' ; involute teeth have the advantage of preserving the velocity ratio of the wheels unchanged for variable distances between the centres of the wheels, and are employed in rolling mills.

When the radius a' is made infinite, the corresponding wheel becomes converted into a *rack*, and the teeth on the rack are shaped by cycloids.

Sometimes to ensure greater regularity of working, *helical* teeth are employed, and now the tooth of one helix on the cylinder of radius a works with the tooth of the helix of equal pitch on the cylinder of radius a' ; so that when one helix rolls on another of equal pitch on a parallel axis, any point of the helix describes an epicycloid. (MacCord, *Kinematics*; G. B. Grant, *Odontics*).

166. *The Double Generation of Epi- and Hypo-cycloids.*

Produce PI both ways to meet the circles O and O' again in H and H' , and draw EPE' parallel to OO' to meet OH and $O'H'$ in E and E' .

Then E and E' are the centres of circles, of radii $a+c$ and $a'-c$, which touch each other at P and the circles O and O' at H and H' ; so that the epicycloid AP and the

hypocycloid $A'P$ can be described by the rolling of these circles on the circles O and O' ; we thus perceive that there is a *double mode of generation of the epicycloid and hypocycloid*.

Relatively to the circle O' , any point on the circumference of the circle O describes an epicycloid; and a hypocycloid if it is made to roll inside the circle O' .

The envelope of the diameter QCP is another epicycloid, described by a rolling circle of radius $\frac{1}{2}c$, as is readily perceived when we notice that the point of contact of the envelope is at the foot of the perpendicular drawn from I on PQ .

The envelope of any other carried straight line, say parallel to PQ , will be a parallel curve to the epicycloid, the envelope of PQ , and will therefore have an epicycloidal evolute.

167. *Epi- and Hypo-Trochoids, or Bicircloids.*

A point fixed in CP at a distance kc from C will describe a curve, given by

$$x = (a + c)\cos \theta + kc \cos(1 + a/c)\theta,$$

$$y = (a + c)\sin \theta + kc \sin(1 + a/c)\theta;$$

these curves are called epi- or hypo-trochoids, and sometimes *bicircloids*.

The relative orbit of two planets is a bicircloid, if the planets describe circles round the Sun; figures of the relative orbits of the Earth and the different planets, drawn mechanically by Mr. Perigal, are given in Proctor's *Geometry of Cycloids*.

For if a, β denote the (mean) distances from the Sun, and n, m the mean motions, the relative orbit is given by

$$x = a \cos nt + \beta \cos mt, \quad y = a \sin nt + \beta \sin mt;$$

where $n^2 a^3 = m^2 \beta^3$, by Kepler's Third Law (§ 179); so that

$$a + c = a, \quad kc = \beta, \quad \text{and} \quad 1 + a/c = m/n = (a/\beta)^{\frac{2}{3}};$$

or
$$a = a - \beta^{\frac{3}{2}}/a^{\frac{1}{2}}, \quad c = \beta^{\frac{3}{2}}/a^{\frac{1}{2}}.$$

When $c = a$, or $m = 2n$, the bicircloids are *limaçons* (§ 156), as is otherwise geometrically obvious. Thus if we assume that the period of Mars is two years, the relative orbit of the Earth and Mars will be a *limaçon*.

Examples.

- (1) Prove that if an epicycloid rolls on a straight line, the centre will describe an ellipse.
- (2) Prove that, if a helix rolls on a straight line, any point on the helix will describe a cycloid.
- (3) The shadow of a helix on a plane perpendicular to its axis, thrown by parallel rays of light, is a trochoid.

Find when the trochoid will be a cycloid.

- (4) The path of a steamer turning uniformly in a current will be a trochoid relatively to the land.
- (5) Show that a variable velocity ratio of two wheels can be attained by cutting teeth in equiangular spirals of the same radial angle.
- (6) If an equiangular spiral rolls on a circle, the pole will describe an involute of a circle.
- (7) Show that the path of the Moon relatively to the Sun has no points of inflexion.

- (8) The isoptic locus of an epicycloid is an epitrochoid (Wolstenholme, *Proc. London Math. Soc.*, vol. iv.).

If the isoptic locus passes through the centre, its equation is of the form $r = b \sin \{a\theta / (a + 2c)\}$.

- (9) Prove that the epicycloid has (i.) one cusp when $c = a, 2a, 3a, 4a, \dots$; and then $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots$; (ii.) two cusps when $2c = a, 3a, 5a, \dots$; and then $m = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots$; and draw them.
- (10) Prove that the equation of the *Tricusp* hypocycloid can be written

- (i.) $p = \frac{1}{3}a \cos 3\omega$, (ii.) $s = \frac{8}{9}a \sin 3\psi$,
 (iii.) $(x^2 + y^2)^2 + \frac{8}{3}ax(x^2 - 3y^2) + 2a^2(x^2 + y^2) - \frac{1}{3}a^4 = 0$,
 (iv.) $r^4 + \frac{8}{3}ar^3 \cos 3\theta + 2a^2r^2 - \frac{1}{3}a^4 = 0$.

Prove that if the tangent at P meets the Tricusp again in Q and R ,

- (i.) the length $QR = \frac{4}{3}a$,
 (ii.) the tangents at Q and R intersect at right angles in a point T on the inscribed circle of the Tricusp,
 (iii.) the normals at P, Q, R intersect in a point N on the circumscribed circle,
 (iv.) NT passes through the centre.

168. *Inversion and Inverse Curves.*

When the vector OP of a curve is produced to Q , so that OQ is inversely proportional to OP , or $OQ = c^2/OP$, then the locus of Q is called an *inverse curve* of the locus of P with respect to the origin O , or with respect to the circle of the centre O and radius c ; and c is then called the *constant of inversion*.

Thus if $u = f\theta$ is the polar equation of the locus of P , then $r = c^2/f\theta$ is the polar equation of the inverse curve, the locus of Q , with respect to the origin O .

The inverse of a circle (or sphere) is another circle (or sphere); except when the circle (or sphere) passes through the centre of inversion, when the inverse is a straight line (or plane).

For if OPQ meets a circle (or sphere) in P and Q , then $OP \cdot OQ$ is constant and equal to OT^2 for an external origin O , where OT is a tangent to the circle (or sphere); so that a circle (or sphere) is its own inverse with respect to any origin O ; and varying the constant of inversion gives a similar curve (or surface); in this case another circle (or sphere).

Maxwell calls the rectangle on the segments of a chord through O made by the circle (or sphere) the *power* of the circle (or sphere) with respect to O ; and now if the distances from O to the centre of the circle or sphere and of its inverse are denoted by a and a' , and the radii by b and b' , then $a^2 - b^2$, $a'^2 - b'^2$ are the *powers* of the circles or spheres, and

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c^2}{a^2 - b^2} = \frac{a'^2 - b'^2}{c^2}.$$

Any curve Pp and its inverse Qq with respect to O will cut the vector OPQ at supplementary radial angles; for since $OP \cdot OQ = c^2 = Op \cdot Oq$, a circle can be drawn through $PQqp$, and therefore the angles pPQ , pqQ are supplementary, and Pp , Qq are ultimately the tangents at P and Q , when p and therefore q are brought up close to P and Q .

Or, otherwise, denoting the radial angles by ϕ and ϕ' ,

$$\cot \phi = d \log r / d\theta = -d \log u / d\theta = -f'\theta / f\theta = -\cot \phi',$$

$$\cot \phi + \cot \phi' = 0, \text{ or } \phi + \phi' = \pi;$$

so that the radial angles are supplementary.

Any two curves and their inverses will therefore cut at the same angle.

A small element of area is consequently unaltered in shape by inversion, but the alteration of linear scale is

$$r'/r = c^2/r^2 = r'^2/c^2.$$

So also in space, an element of volume will be undistorted by inversion, but will be changed in volume in the scale c^6/r^6 or r'^6/c^6 .

If accented letters refer generally to the inverse curve, then

$$Ot' = c^2/Og, \quad Og' = c^2/Ot;$$

$$p' = c^2/Pg, \quad p = c^2/Qg';$$

$$Pt \cdot Qt' = c^2 \sec^2 \phi;$$

$$\frac{r'}{\rho'} + \frac{r}{\rho} = \frac{dp}{dr} + \frac{dp'}{dr'} = \frac{dr \sin \phi}{dr} + \frac{dr' \sin \phi}{dr'} = 2 \sin \phi;$$

and the circle of curvature obviously inverts into the circle of curvature of the new curve.

Lines of curvature (§ 130) will correspond on a surface and its inverse; and if a circle of principal curvature is drawn, the normal plane of this circle and the sphere through the circle and origin of inversion O will invert into the sphere and plane through the corresponding circle of principal curvature.

When the Cartesian equation of a plane curve is given in rectangular coordinates x and y , the equation of the inverse curve with respect to the origin O is obtained by writing $c^2x/(x^2+y^2)$ for x , and $c^2y/(x^2+y^2)$ for y .

Thus the inverse of the parabola $y^2 = px$ with respect to the vertex is, writing a for c^2/p , the *cissoïd*

$$x(x^2 + y^2) = ay^2, \text{ or } y^2 = x^3/(a - x);$$

or in polar coordinates,

$$r = a(\sec \theta - \cos \theta),$$

the locus of t in a circle, for an origin on the circumference (ex. 11, p. 42).

A jointed parallelogram of rods, $FGHK$, is taken, and the longer rods are crossed; any fixed point O in one of the rods FG is taken, and $OPQR$ is drawn parallel to FH or GK , to meet FK in P , GH in Q , and HK in R (fig. 51).

Then P , Q , R are also fixed points in the bars, such that when again opened into a parallelogram, $OPRQ$ will form another parallelogram, the sides of which are parallel to the diagonals of $FGHK$, and the area of which bears a constant ratio to the area of $FGHK$; so that $OP \cdot OQ$ is constant; and thus the parallelogram $FGHK$, when crossed, will act as an inverter, P and Q describing inverse curves when O is fixed.

We can show Hart's parallel motion working in conjunction with Peaucellier's by drawing FG parallel to PM or LQ , GQH parallel to OL , and FPK parallel to OM ; and now O , P , R , Q are the middle points of the bars of the crossed parallelogram $FGHK$.

We may join LR and MR by bars, and now the two rhombuses $LPMQ$, $OLRM$ are said to make a complete Peaucellier cell.

When P and Q are inside the cell, the cell is called *positive* (fig. 51), where it is shown acting as the parallel motion of a beam engine.

But when P and Q are outside, it is called a *negative* cell (fig. 52); and now it forms a compact mechanism for drawing not only straight lines, but circles of very large radius, as required in Architecture.

(Kempe, *How to Draw a Straight Line.*)

While Q describes a straight line and P a circular arc, the point R will describe a curve whose polar equation is

of the form
$$r = a \sec \theta \pm b \cos \theta,$$

the inverse of a conic section with respect to a vertex.

The curves described by any point Q on the connecting rod AP in three bar motion have been studied by Roberts, Cayley, and Clifford; any such curve being capable of a triple generation by means of bars (Clifford, *Kinematic*).

When a crossed parallelogram is employed, as in figs. 51, 52, and in fig. 49, the relative motion of the short bars FG and HK is imitated by rolling an ellipse with foci F, G , on an equal ellipse with foci H, K ; and the relative motion of the long bars GH, FK is imitated by rolling a hyperbola on an equal hyperbola; and relatively to one bar any point carried by the opposite bar will describe the pedal of an ellipse or hyperbola.

170. Polar Reciprocals.

The inverse of the pedal of a curve with respect to the same pole is called a *polar reciprocal* of the curve.

For instance, the pedal of a circle with respect to any point O is the limaçon (§ 156)

$$r = a + b \cos \theta;$$

and therefore the polar reciprocal of a circle is the inverse of a limaçon, and its equation is

$$r = c^2 / (a + b \cos \theta),$$

the polar equation of a conic section, with a focus at the pole, excentricity $e = b/a$, and semi-latus-rectum $l = c^2/a$.

Again, the polar reciprocal of an epicycloid with respect to the centre O is $r \cos m\theta = b$, a Cotes's spiral (§ 164).

With the usual notation of $\theta, r, \phi, p, \rho, \dots$ for the original curve, and $\theta', r', \phi', \dots$ for the polar reciprocal with respect to the origin, then from §§ 155, 168,

$$\phi + \phi' = \pi, \quad r' = c^2/p, \quad p' = c^2/r;$$

and

$$\rho \rho' = r \frac{dr}{dp} \cdot r' \frac{dr'}{dp'} = c^2 \frac{r^3}{p^3} = c^2 \operatorname{cosec}^3 \phi.$$

171. *Orthogonal and Oblique Trajectories.*

A curve cutting at right angles a system of curves is called an *orthogonal trajectory* of the system; and a curve cutting the system at any constant angle γ other than a right angle is called an *oblique trajectory*.

In § 101, it was shown that the orthogonal trajectories of the cycloids described by all the points on the circumference of a wheel are equal cycloids.

Familiar instances of orthogonal trajectories are seen with (i.) horizontal and vertical straight lines, (ii.) straight lines radiating from a centre, and the concentric circles, (iii.) confocal and coaxial parabolas, (iv.) rectangular hyperbolas, the asymptotes of one system being the axes of the other system, (v.) confocal ellipses and hyperbolas; while the oblique trajectories of system (i.) are inclined straight lines, of (ii.) are equiangular spirals, of (iii.) are confocal but not coaxial parabolas, and of (iv.) are concentric, but not coaxial rectangular hyperbolas.

Since any two curves and the corresponding *inverse* curves cut at the same angle, it follows that the inverse of a system of orthogonal or oblique trajectories forms a new system of orthogonal or oblique trajectories.

We have seen (§ 133) that the graphs of the conjugate functions u and v , given by $u+iv=f(x+iy)$, when u or v is equated to a constant, form an orthogonal system.

Also, since in an oblique trajectory cutting the curves u and v (§ 133) at constant angles γ and $\frac{1}{2}\pi - \gamma$

$$\begin{aligned} & ds_1 \cos \gamma + ds_2 \sin \gamma = 0, \\ \text{and} \quad & ds_1/du = ds_2/dv = J^{-\frac{1}{2}}; \end{aligned}$$

$$\text{therefore} \quad du \cos \gamma + dv \sin \gamma = 0,$$

$$\text{or} \quad u \cos \gamma + v \sin \gamma = \text{a constant},$$

is the general equation of an oblique trajectory.

Mr. Larmor has shown (*Proc. London Math. Soc.*, vol. xv.) that in a field of force (§ 83), of which the potential is J , a particle will describe an oblique trajectory with velocity V , if started so that $\frac{1}{2}V^2 = J$; or that a ray of light will describe this oblique trajectory if the refractive index varies as $J^{-\frac{1}{2}}$.

Then, if R denotes the radius of curvature of the trajectory, resolving normally,

$$\frac{V^2}{R} = \frac{2J}{R} = \frac{\partial J}{\partial s_2} \cos \gamma + \frac{\partial J}{\partial s_1} \sin \gamma = J^{\frac{1}{2}} \left(\frac{\partial J}{\partial u} \cos \gamma + \frac{\partial J}{\partial v} \sin \gamma \right),$$

or
$$\frac{1}{R} = \frac{\partial J^{\frac{1}{2}}}{\partial u} \cos \gamma + \frac{\partial J^{\frac{1}{2}}}{\partial v} \sin \gamma.$$

Any small contour in the (x, y) diagram will be transformed into a similar contour in the (u, v) diagram, since angles are unaltered in this transformation, $J^{\frac{1}{2}}$ denoting the scale of the transformation, or the ratio of the linear dimensions of corresponding elements; this is the condition to be satisfied in maps and charts.

Thus, for instance, if

$$u + iv = (x + iy)^n = r^n \cos n\theta + ir^n \sin n\theta,$$

by De Moivre's Theorem (§ 111), then, on putting

$$u + iv = c^{n-1}(x' + iy'),$$

$$u = c^{n-1}x' = r^n \cos n\theta, \quad v = c^{n-1}y' = r^n \sin n\theta;$$

and a new system of curves is obtained in which r is changed into r^n/c^{n-1} and θ into $n\theta$; so that we may put

$$r' = r^n/c^{n-1}, \quad \theta' = n\theta;$$

and now the radial angle ϕ is the same in the old and the new system; and

$$n \frac{r'}{\rho} - \frac{r}{\rho} = n \frac{dr' \sin \phi}{dr'} - \frac{dr \sin \phi}{dr} = (n-1) \sin \phi.$$

The case of $n = -1$ is equivalent to the operation of *inversion* (§ 168) combined with *reflexion* in Ox .

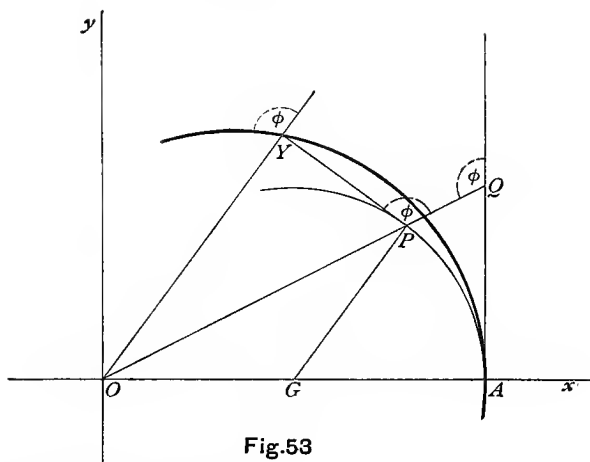


Fig.53

172. *Critical Orbits.*

The orthogonal trajectories given by

$$u = r^n \cos n\theta, \quad v = r^n \sin n\theta,$$

are of important interest; the oblique trajectories being

$$u \cos \gamma + v \sin \gamma = r^n \cos(n\theta - \gamma) = a \text{ constant.}$$

These curves are called *Critical Orbits* by Clifford (*Kinematic*, p. 113) because they are described as orbits under a central force varying as some power, $2n-3$, of r (§ 84), while the velocity varies as some other power, $n-1$, of r . They are also *catenaries*, the curves assumed by a uniform chain under a central force varying as the power $n-2$ of r .

Writing the equations,

$$r^n \cos n\theta = a^n, \quad r^n \sin n\theta = b^n, \quad r^n \cos(n\theta - \gamma) = c^n;$$

then (i.) $n = \pm 1$ gives a system of straight lines and circles, (ii.) $n = \pm \frac{1}{2}$ a system of confocal parabolas and cardioids, (iii.) $n = \pm 2$ a system of rectangular hyperbolas and *lemniscates*.

Changing n into $-n$ gives the inverse system of curves

$$r^n = a^n \cos n\theta, \quad r^n = b^n \sin n\theta, \quad r^n = c^n \cos(n\theta - \gamma);$$

and now, in $r^n = a^n \cos n\theta$,

$$\cot \phi = d \log r / d\theta = -\tan n\theta = \cot(\frac{1}{2}\pi + n\theta),$$

or $\phi = \frac{1}{2}\pi + \frac{1}{2}n\theta$; so that (fig. 53)

$$POY = n\theta, \text{ and } AOY = \omega = (n+1)\theta.$$

Then $p = r \cos n\theta = r^{n+1}/a^n$,

or $p = a(\cos n\theta)^{1+1/n} = a \left(\cos \frac{n\omega}{n+1} \right)^{\frac{n+1}{n}},$

$$p^m = a^m \cos m\omega, \text{ where } m = n/(n+1);$$

so that the pedal curve AY of the critical orbit AP is also a critical orbit; and the equation of the polar reciprocal of AP may be written

$$r^m \cos m\theta = a^m,$$

another critical orbit.

Consider the path of a ray of light in the Earth's atmosphere, on Simpson's assumption that the refraction index μ varies inversely as the $n+1^{\text{th}}$ power of the distance r from the centre of the Earth.

Then (§ 73) $\mu \sin \phi$ or μp is constant along the ray, or p varies as r^{n+1} , so that the path of the ray is either the critical orbit $r^n = a^n \cos n\theta$, or an oblique trajectory.

Again, in these curves, the chord of curvature through O

$$= 2pdr/dp = 2r/(n+1),$$

so that (fig. 10) $\rho = Pg/(n+1).$

173. *Roulettes of Critical Orbits.*

When AP (fig. 48) is the roulette of the pole P of the critical orbit $r^n = a^n \cos n\theta$ with respect to the straight line Ox , then denoting the angle IPM by ψ , and MP by y ,

$$\psi = n\theta = m\omega,$$

and $y^m = p^m = a^m \cos m\omega = a^m \cos \psi$,

the relation connecting y and ψ in the roulette.

Differentiating logarithmically with respect to s ,

$$\frac{m}{y} \frac{dy}{ds} = -\tan \psi \frac{d\psi}{ds};$$

and since
$$\frac{dy}{ds} = -\sin \psi, \quad \frac{ds}{d\psi} = \rho,$$

therefore
$$\rho = y \sec \psi / m = PI / m,$$

so that ρ the radius of curvature of the roulette is $1/m$ or $1 + 1/n$ times the length of the normal PI , and therefore $m/(m+1)$ times the radius of curvature of the rolling pedal $A'P$; and the evolute of the roulette will possess a similar property.

For instance, if $n=1$, the rolling curve is a circle, and the roulette is a cycloid; and then $\rho=2PI$ (§ 101), and the evolute of AP is an equal cycloid.

As applications of these roulettes AP , we may instance

(i.) that AP is a catenary curve with Ox horizontal, when the line density is made proportional to the $2 + 1/n^{\text{th}}$ power of $\sec \psi$, or of the tension; reducing, for $n = -\frac{1}{2}$, to the ordinary catenary, which is therefore the roulette of the focus of a parabola;

(ii.) that, with m negative and $= -p$, AP can be described as a trajectory by a projectile in a resisting medium, in which the retardation due to the resistance is

$$g\left(1 - \frac{1}{2p}\right)\left\{1 - \left(\frac{v_0}{v}\right)^{4p}\right\}^{\frac{1}{2}},$$

where v denotes the velocity at P , and v_0 at A ; reducing, for $p = \frac{1}{2}$, to the ordinary parabolic trajectory, the roulette of the pole of $r^{\frac{1}{3}} \cos \frac{1}{3}\theta = a^{\frac{1}{3}}$, the first negative pedal of the parabola $r^{\frac{1}{2}} \cos \frac{1}{2}\theta = a^{\frac{1}{2}}$;

(iii.) that AP is also the path of a ray of light, when the refractive index varies as y^p ; a catenary for $p=1$.

174. *Vectors and Vectorial Equations.*

The quantity $z = x + iy$ is now called the *vector* or *step* from O to the point (x, y) ; being composed of a step of length x parallel to Ox and then a step of length y parallel to Oy , according to Argand's representation.

Then if $z' = x' + iy'$ represents another vector,

$$z - z' = x - x' + i \cdot y - y'$$

represents the vector or step from (x', y') to (x, y) .

It is convenient in physical application to consider the logarithm of the vector, and now

$$w = \log(z - z')$$

$$\text{or } u + iv = \log(x - x' + i \cdot y - y')$$

$$= \log \sqrt{\{(x - x')^2 + (y - y')^2\}} + i \tan\{(y - y')/(x - x')\}$$

gives u the velocity function, and v the stream function at (x, y) of a *source* or *electrode* at (x', y') ; and we may call w the *vector potential*.

Suppose for instance we place n equal positive electrodes at points $A_0, A_1 \dots A_{n-1}$, equally spaced on a circle of radius a ; the vector of the point A_s being

$$a \cos 2s\pi/n + ia \sin 2s\pi/n = a \exp(2is\pi/n),$$

the vector potential

$$w = \log \prod_{s=0}^{s=n-1} \{z - a \exp(2is\pi/n)\} = \log(z^n - a^n);$$

and now, with polar coordinates,

$$z = r \cos \theta + ir \sin \theta = re^{i\theta},$$

$$z^n = r^n e^{in\theta} = r^n \cos n\theta + ir^n \sin n\theta;$$

$$w = \log(r^n \cos n\theta - a^n + ir^n \sin n\theta),$$

$$u + iv = \log \sqrt{(r^{2n} - 2a^n r^n \cos n\theta + a^{2n})}$$

$$+ i \tan^{-1}\{r^n \sin n\theta / (r^n \cos n\theta - a^n)\};$$

$$\text{or } r^{2n} - 2a^n r^n \cos n\theta + a^{2n} = e^{2u},$$

$$r^n \sin(v - n\theta) = a^n \sin v,$$

representing an orthogonal system of curves.

Denoting by r_s, θ_s the polar coordinates of a point P relatively to the origin A_s , then

$$u = \log r_0 r_1 r_2 \dots r_{n-1}, \quad v = \theta_0 + \theta_1 + \theta_2 + \dots + \theta_{n-1},$$

giving De Moivre's and Cotes's properties of the circle.

Suppose for instance $n=2$, then

$$u = \log r_1 r_2, \text{ or } r^4 - 2a^2 r^2 \cos 2\theta + a^4 = e^{2u};$$

$$\text{and } v = \theta_0 + \theta_1, \text{ or } r^2 \sin(2\theta - v) = -a^2 \sin v,$$

representing *Cassinians*, and the orthogonal rectangular hyperbolas, passing through A_0 and A_1 .

When n equal negative electrodes are placed on the circle, at $B_1, B_2, \dots B_n$, midway between the positive electrodes, the vector of B_s being

$$a \exp\{(2s-1)\pi/n\},$$

then their vector potential is

$$\log \prod_{s=1}^{s=n} \{z - a \exp(2s-1)\pi/n\} = \log(z^n + a^n);$$

so that for the system of positive and negative electrodes

$$w = \log(z^n - a^n)/(z^n + a^n)$$

$$= \frac{1}{2} \log \frac{r^{2n} - 2a^n r^n \cos n\theta + a^{2n}}{r^{2n} + 2a^n r^n \cos n\theta + a^{2n}}$$

$$+ i \tan^{-1} \frac{r^n \sin n\theta}{r^n \cos n\theta - a^n} - i \tan^{-1} \frac{r^n \sin n\theta}{r^n \cos n\theta + a^n}$$

$$= -\tanh^{-1} \frac{2a^n r^n \cos n\theta}{r^{2n} + a^{2n}} - i \tanh^{-1} \frac{2a^n r^n \sin n\theta}{r^{2n} - a^{2n}},$$

giving another orthogonal system of curves,

$$r^{2n} + 2a^n r^n \cos n\theta \coth u + a^{2n} = 0,$$

$$r^{2n} + 2a^n r^n \sin n\theta \cot v - a^{2n} = 0.$$

When $n=1$, these curves are circles, the *dipolar circles* of the next article.

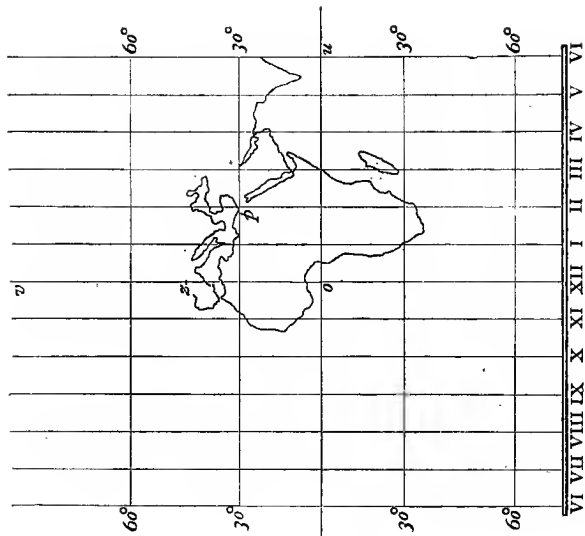
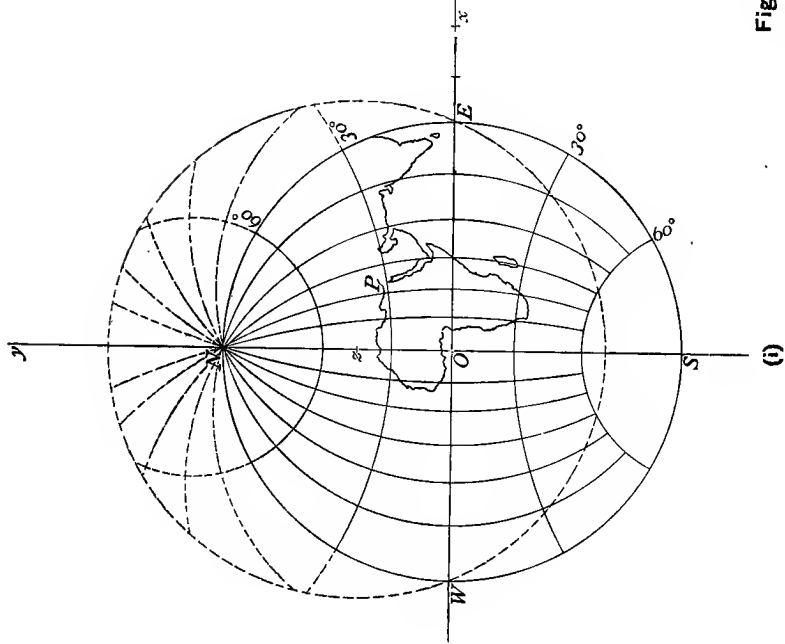


Fig. 54

175. *Dipolar Circles.*

When we put $x+iy=c \tan \frac{1}{2}(u+iv)$, and write it

$$x+iy=c \frac{\sin \frac{1}{2}(u+iv) \cos \frac{1}{2}(u-iv)}{\cos \frac{1}{2}(u+iv) \cos \frac{1}{2}(u-iv)} = c \frac{\sin u + i \sinh v}{\cos u + \cosh v},$$

then
$$x = \frac{c \sin u}{\cosh v + \cos u}, \quad y = \frac{c \sinh v}{\cosh v + \cos u},$$

thus dividing $\tan \frac{1}{2}(u+iv)$ into a real and imaginary part.

Also
$$\begin{aligned} \tan u &= \tan \left\{ \frac{1}{2}(u+iv) + \frac{1}{2}(u-iv) \right\} \\ &= c \frac{x+iy+x-iy}{c^2-x^2-y^2} = \frac{2cx}{c^2-x^2-y^2}, \end{aligned}$$

and
$$\begin{aligned} \tan iv &= \tan \left\{ \frac{1}{2}(u+iv) - \frac{1}{2}(u-iv) \right\} \\ &= c \frac{x+iy-x+iy}{c^2+x^2+y^2} = \frac{2icy}{c^2+x^2+y^2}; \end{aligned}$$

so that
$$x^2 + y^2 + 2cx \cot u - c^2 = 0 \dots\dots\dots (i.),$$

$$x^2 + y^2 - 2cy \coth v + c^2 = 0 \dots\dots\dots (ii.),$$

the equations of a system of orthogonal dipolar circles (fig. 54.i.) ; called *dipolar* because the system (i.) represents circles passing through two poles N and S, while system (ii.) represents orthogonal circles having the same radical axis Ox ; thus representing the meridians and parallels in the stereographic projection of the terrestrial globe.

Putting u or $v=nt$, we obtain circles described under excentric centres of force ; and writing them

$$(x+c \cot u)^2 + y^2 = c^2 \operatorname{cosec}^2 u \dots\dots\dots (i.)$$

$$x^2 + (y-c \coth v)^2 = c^2 \operatorname{cosech}^2 v \dots\dots\dots (ii.),$$

then (i.) represents a circle, centre $(-c \cot u, 0)$, and radius $c \operatorname{cosec} u$; and (ii.) a circle, centre $(0, c \coth v)$ and radius $c \operatorname{cosech} v$; and with respect to O , the *power* of (i.) is $-c^2$, and of (ii.) is c^2 .

At any point P , the angle $NPS = \pi - u$, while the ratio $SP/NP = e^v$, or $v = \log(SP/NP)$.

If λ denotes the latitude of any parallel of latitude circle (ii.), then since on the stereographic projection the radius of this circle is $c \cot \lambda$, therefore (§ 34)

$$\cot \lambda = \operatorname{cosech} v, \tan \lambda = \sinh v, \sin \lambda = \tanh v,$$

$$\cos \lambda = \operatorname{sech} v, \tan \frac{1}{2} \lambda = \tanh \frac{1}{2} v, \text{ or } \lambda = \operatorname{gd} v,$$

$$v = \log(\sec \lambda + \tan \lambda) = \log \tan\left(\frac{1}{4} \pi + \frac{1}{2} \lambda\right).$$

while $\lambda = \operatorname{gd}(u \cot \gamma)$ represents a *loxodrome* or *rhumb line*, cutting the meridians at a constant angle γ .

The inverse system of curves with respect to a pole N or S will be radiating straight lines and concentric circles, representing meridians and parallels in the neighbourhood of a pole, when projected stereographically from the other pole, as in Godfray's Great Circle Chart.

A *great circle* on this chart will be represented by a straight line; and a *rhumb line*, that is an oblique trajectory of the meridians and parallels, will be an equiangular spiral; so that on the stereographic projection it will be an inverse of an equiangular spiral.

The general operation of inversion with respect to any pole, combined with a change of origin, is represented by the *linear substitution* (Schwarz)

$$z' = (az + b)/(Az + B),$$

by which a polygon in the z plane, bounded by straight lines or circles, is transformed into a polygon bounded by circles, cutting at the same angles, in the z' plane.

In Godfray's Great Circle Chart we transform to

$$z' = x' + iy' = c \exp(i\alpha - \beta) = ce^{-\beta}(\cos \alpha + i \sin \alpha),$$

so that α is the longitude, and the latitude λ is given by

$$r' = ce^{-\beta} = c \tan\left(\frac{1}{4} \pi - \frac{1}{2} \lambda\right),$$

or

$$\beta = \log(\sec \lambda + \tan \lambda), \lambda = \operatorname{gd} \beta.$$

176. *Mercator's Chart.*

Draw any contour on the (u, v) diagram (fig. 54, ii.) corresponding to a contour, the outline of a country, in the (x, y) stereographic diagram; we thus obtain a new map, called Mercator's projection, in which the meridians and parallels are orthogonal systems of parallel straight lines; and while a length u represents the longitude, a length $v = \log(\sec \lambda + \tan \lambda)$ will represent the latitude λ ; also $dv/d\lambda = \sec \lambda$, so that the minute of latitude in Mercator's chart increases as the secant of the latitude, being equal to the minute of longitude only at the equator.

Twenty-four standard meridians at equal intervals of 15° in longitude from Greenwich mark the *standard time* at a place; the standard time being the mean solar time at the nearest standard meridian.

Taking the Godfray Chart as representing any system of parallels and meridians with respect to an axis through any zenith Z , say in longitude 0 and latitude $\text{gd } b$, then the representation on Mercator's Chart will be given by

$$\tan \frac{1}{2}(u + i \cdot v - b) = \exp(i\alpha - \beta) = x' + iy' = z';$$

or, dropping b as representing a mere change of origin,

$$\begin{aligned} i\alpha - \beta &= \log \tan \frac{1}{2}(u + iv) \\ &= \log \frac{\sin u + i \sinh v}{\cos u + \cosh v} = i \tan^{-1} \frac{\sinh v}{\sin u} - \coth^{-1} \frac{\cosh v}{\cos u}; \end{aligned}$$

so that $\sinh v = \tan \alpha \sin u$, $\cosh v = \coth \beta \cos u$;

the representation on Mercator's Chart of a system of parallel small circles and their meridians; these lines are called *Sumner lines* (p. 203), being the lines on Mercator's Chart on which a celestial body in the zenith at Z will at any instant have the same altitude.

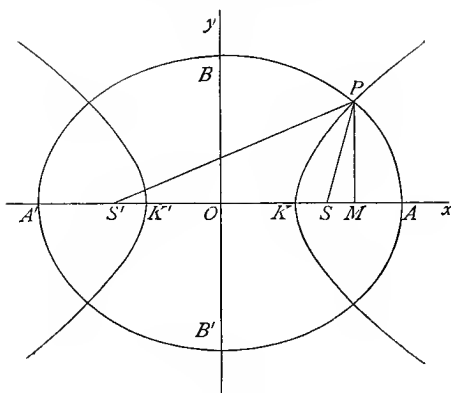


Fig. 55

177. *Confocal Ellipses and Hyperbolas.*

As another practical illustration of conjugate functions, put

$$x + iy = c \cosh(u + iv) \\ = c(\cosh u \cos v + i \sinh u \sin v);$$

then $x = c \cosh u \cos v$, $y = c \sinh u \sin v$;

and alternately eliminating u and v ,

$$\frac{x^2}{c^2 \cosh^2 u} + \frac{y^2}{c^2 \sinh^2 u} = 1 \dots \dots \dots (i.)$$

$$\frac{x^2}{c^2 \cos^2 v} - \frac{y^2}{c^2 \sin^2 v} = 1 \dots \dots \dots (ii.)$$

representing a system of confocal ellipses and hyperbolas for constant values of u and v the polar reciprocals with respect to N or S of the circles v of § 175; and $\operatorname{sech} u$ will be the excentricity of the ellipse, $\sec v$ of the hyperbola (fig. 55); while (§ 35) v will be the excentric angle or excentric anomaly of a point P on the ellipse AP , and u will be hyperbolic excentric anomaly of the point P on the hyperbola KP .

Put $v = nt$, then

$$x + iy = c \cosh (u + int) = c \cos (nt - iv)$$

represents confocal ellipses described under a central attraction to O , varying as the distance, the period being $2\pi/n$; while hyperbolas are described with $u = nt$, and $x + iy = c \cosh (nt + iv)$.

In an oblique trajectory

$$u \cos \gamma + v \sin \gamma = \text{constant},$$

we may put $v = n(t - \tau) \cos \gamma$, $u = -n(t - \tau') \sin \gamma$;

so that $x + iy = c \cos n\{(t - \tau) \cos \gamma + i(t - \tau') \sin \gamma\}$;

and the equation is obtained by eliminating t between

$$x = c \cos \{n(t - \tau) \cos \gamma\} \cosh \{n(t - \tau') \sin \gamma\},$$

$$y = -c \sin \{n(t - \tau) \cos \gamma\} \sinh \{n(t - \tau') \sin \gamma\}.$$

The equations of the tangent and normal of the ellipse at P , which are also the normal and tangent of the hyperbola at P , are

$$x \frac{\cos v}{\cosh u} + y \frac{\sin v}{\sinh u} = c \dots\dots\dots (\text{iii.}),$$

$$x \frac{\cosh u}{\cos v} - y \frac{\sinh u}{\sin v} = c \dots\dots\dots (\text{iv.}).$$

Denoting by p and q the lengths of the perpendiculars from O upon the tangents at P of the ellipse and hyperbola,

$$p = c \cosh u \sinh u / (\cosh^2 u - \cos^2 v)^{\frac{1}{2}},$$

$$q = c \sin v \cos v / (\cosh^2 u - \cos^2 v)^{\frac{1}{2}};$$

The pole of (iii.) with respect to the hyperbola (ii.) will be $(c \cos^3 v / \cosh u, -c \sin^3 v / \sinh u)$, and this pole will therefore be Q , the centre of curvature of the ellipse at P (§ 98); and $PQ = \rho = c(\cosh^2 u - \cos^2 v)^{\frac{3}{2}} / (\cosh u \sinh u)$.

Similarly $(c \cosh^3 u / \cos v, -c \sinh^3 u / \sin v)$, the pole of (iv.) with respect to (i.), is the centre of curvature of the hyperbola at P ; and the radius of curvature is

$$c(\cosh^2 u - \cos^2 v)^{\frac{3}{2}} / (\cos v \sin v).$$

We may consider $w = u + iv$ as representing a vector of curvilinear steps u and v ; u along a hyperbola and v along a confocal ellipse, meeting at the point

$$z = x + iy = c \cosh(u + iv).$$

If P' is any other point given by

$$z' = x' + iy' = c \cosh(u' + iv'),$$

then the vector PP' is given by

$$\begin{aligned} z - z' &= c \{ \cosh(u + iv) - \cosh(u' + iv') \} \\ &= 2c \sinh \frac{1}{2}(u + u' + i.v + v') \sinh \frac{1}{2}(u - u' + i.v - v'); \end{aligned}$$

so that PP'^2

$$\begin{aligned} &= 4c^2 \sinh \frac{1}{2}(u + u' + i.v + v') \sinh \frac{1}{2}(u - u' + i.v - v') \\ &\quad \sinh \frac{1}{2}(u + u' - i.v + v') \sinh \frac{1}{2}(u - u' - i.v - v') \\ &= c^2 \{ \cosh(u + u') - \cos(v + v') \} \{ \cosh(u - u') - \cos(v - v') \}. \end{aligned}$$

This expression is unaltered by an interchange of u, u' , or v, v' ; so that if the points p, p' , called *corresponding points*, are given by (u, v') , (u', v) , then $pp' = PP'$.

Further, if ϕ denotes the angle between PP' and pp' ,

$$\begin{aligned} \text{then } i\phi &= \log \frac{\cosh(u + iv) - \cosh(u' + iv')}{\cosh(u + iv') - \cosh(u' + iv)} \\ &= \log \frac{\tanh \frac{1}{2}(u - u') + i \tanh \frac{1}{2}(v - v')}{\tanh \frac{1}{2}(u - u') - i \tanh \frac{1}{2}(v - v')} \\ &= 2i \tan^{-1} \frac{\tanh \frac{1}{2}(v - v')}{\tanh \frac{1}{2}(u - u')}. \end{aligned}$$

Captain Weir's Azimuth Diagram consists of a series of confocals, the hyperbolas marking the longitude or hour angle v , while the ellipses are marked with the degrees in $\text{gd } u$, to represent latitude or declination.

Now, if the hour angle is the complement of v , and if the latitude and declination are $\text{gd } u$ and $\text{gd } d$, a straight line joining the points (u, v) and $(d, 0)$ will give the azimuth or bearing of the object from the meridian Oy .

(Godfray, *Astronomy*, § 222.)

Changing to the focus S or S' as origin

$x - c + iy = 2c \sinh^2 \frac{1}{2}(u + iv)$, $x + c + iy = 2c \cosh^2 \frac{1}{2}(u + iv)$,
so that if r, r' denote the focal distances $SP, S'P$,

$$\begin{aligned} r &= 2c \sinh \frac{1}{2}(u + iv) \sinh \frac{1}{2}(u - iv) \\ &= c(\cosh u - \cos v) = a(1 - e \cos v); \\ r' &= c(\cosh u + \cos v) = a(1 + e \cos v). \end{aligned}$$

Then $r' + r = 2c \cosh u = AA'$ (v.),
 $r' - r = 2c \cos v = KK'$ (vi.);

and hence $SP = AK, S'P = KA'$; also $OP = KB$.

Denoting the angles $ASP, AS'P$ by θ, θ' , then in Astronomy θ is called the *true anomaly* of P , reckoned from *perihelion* at A , the Sun being supposed to be at the focus S ; but θ' is the true anomaly from *aphelion* A , with the Sun now at the other focus S' .

Then

$$\cos \theta = \frac{x - c}{r} = \frac{\cosh u \cos v - 1}{\cosh u - \cos v}, \quad \cos \theta' = \frac{\cosh u \cos v + 1}{\cosh u + \cos v};$$

$$\text{or} \quad \sin \theta = \frac{\sinh u \sin v}{\cosh u - \cos v}, \quad \sin \theta' = \frac{\sinh u \sin v}{\cosh u + \cos v};$$

$$\tan \frac{1}{2}\theta = \coth \frac{1}{2}u \tan \frac{1}{2}v, \quad \tan \frac{1}{2}\theta' = \tanh \frac{1}{2}u \tan \frac{1}{2}v;$$

$$\text{and} \quad r = SA \cos^2 \frac{1}{2}v \sec^2 \frac{1}{2}\theta, \quad r' = SA' \cos^2 \frac{1}{2}v \sec^2 \frac{1}{2}\theta'.$$

$$\text{Again, since} \quad \cos v = \frac{1 + \cos \theta \cosh u}{\cosh u + \cos \theta} = \frac{e + \cos \theta}{1 + e \cos \theta},$$

$$\text{or} \quad (1 + e \cos \theta)(1 - e \cos v) = 1 - e^2;$$

$$\text{therefore} \quad r = c \sinh^2 u / (\cosh u + \cos \theta),$$

of the form $r = l / (1 + e \cos \theta)$, with $l = c \sinh^2 u / \cosh u$,
the polar equation of the ellipse, with origin at S .

$$\text{Similarly} \quad r' = c \sinh^2 u / (\cosh u - \cos \theta')$$

is the polar equation of the ellipse, with origin at S' ;
while for the hyperbola the corresponding polar equations

$$\text{are} \quad r = \frac{c \sin^2 v}{\cos v - \cos \theta}, \quad r' = \frac{c \sin^2 v}{\cos \theta' - \cos v}.$$

178. *Confocal Limaçons.*

By inversion of this system of conics with respect to either focus, say S , taking $2c$ as the constant of inversion,

$$x + iy = 2c \operatorname{cosech}^2 \frac{1}{2}(u + iv),$$

$$x + 2c + iy = 2c \coth^2 \frac{1}{2}(u + iv);$$

and $r \sinh^2 u = 4c(\cosh u + \cos \theta) \dots\dots\dots (\text{vii.}),$

$$r \sin^2 v = 4c(\cos v - \cos \theta) \dots\dots\dots (\text{viii.}),$$

are the inverses of the ellipse (i.) and hyperbola (ii.), and the pedals with respect to N or S of the circles v of § 175, a system of confocal limaçons (§ 155).

Denoting now by r' the distance from the other focus S' , the position of which is unaltered by inversion, then

$$r' = 2c \coth \frac{1}{2}(u + iv) \coth \frac{1}{2}(u - iv) = 2c \frac{\cosh u + \cos v}{\cosh u - \cos v},$$

while

$$r = 2c \operatorname{cosech} \frac{1}{2}(u + iv) \operatorname{cosech} \frac{1}{2}(u - iv) = \frac{4c}{\cosh u - \cos v};$$

so that $r' + 2c = \frac{4c \cosh u}{\cosh u - \cos v} = r \cosh u \dots\dots\dots (\text{ix.}),$

$$r' - 2c = \frac{4c \cos v}{\cosh u - \cos v} = r \cos v \dots\dots\dots (\text{x.}),$$

equations of the form $r' - lr = \text{constant}$, which are *Cartesian ovals* in the general case, the inverse of a conic with respect to any point on the transverse axis.

The limaçon in which $v = \frac{1}{3}\pi$, $l = \frac{1}{2}$, is called the *Trisectrix*.

For if $A'OR$ is any angle to be trisected (fig. 47, ii.), and if $A'R$ is joined cutting the curve in p ; then, if $A'Op = \theta$,

$$Op = OA'(2 \cos \theta - 1); \text{ and } A'p = 2OA' \sin \frac{1}{2}\theta = A'q,$$

if Op meets the arc $A'R$ in q .

Therefore $\theta = qpA' - OA'p = pqA' - OA'p = OA'q - OA'p = pA'q = \frac{1}{2}qOR$; so that Op is a trisector of the angle $A'OR$.

179. *Kepler's Laws of Planetary Motion.*

Kepler, by long continued observations and measurement of the Sun's diameter and motion in longitude, noticed that

- (i.) the variation of the Sun's apparent diameter d could be expressed by the formula $D(1+e \cos \theta)$, where D denotes the mean angular diameter (about $32'$) and θ the Sun's longitude from perihelion, e being a small constant, about $1/60$;
- (ii.) that the Sun's daily motion in longitude was proportional to the apparent area or square of the diameter.

Since the apparent diameter is inversely proportional to the distance, he deduced the laws called Kepler's Laws—

- (i.) that the relative orbit of the Earth (or any planet) and of the Sun is an ellipse, with a focus at the Sun, if the Sun is supposed fixed, given in polar coordinates by $l/r = d/D = 1 + e \cos \theta$.
- (ii.) that $r^2 d\theta/dt = h$ is constant, and that the Earth therefore sweeps out by its radius vector from the Sun equal areas in equal times.

The third law of Kepler—(iii.) the squares of the periodic times of the planets round the Sun are proportional to the cubes of their mean distances—was easily inferred by arithmetical calculation, when once the distances of the planets from the Sun were measured, in terms of the Sun's distance from the Earth.

Newton inferred from law (ii.) that the earth is attracted by the Sun; and from law (i.) that the attraction must be inversely proportional to the square of the distance; while law (iii.) showed that the attraction of the Sun was of the same nature on all the planets (§ 84).

By Newton's Law of Universal Gravitation, employing c.g.s. units (§ 142), the attraction between two spherical bodies, for instance, the Sun and the Earth, weighing S and E grammes, when their centres are a centimetres apart, will be given by the expression $CSEa^{-2}$ (*dynes*); and C is called the *constant of gravitation*, being the attraction in dynes between two spheres, each weighing one gramme, when their centres are one centimetre apart.

Then Kepler's Third Law, in a mathematical form, asserts that if T is the period in seconds of the Sun and planet, and if $n = 2\pi/T$ denotes the *mean motion*, then by ex. 1, p. 175, $n^2a^3 = 4\pi^2a^3/T^2 = C(S + E)$.

According to the Cavendish experiment, now being repeated with improved apparatus by Mr. C. V. Boys, we may take $C = 10^{-8} \times 6.48$, $1/C = 10^7 \times 1.54$.

(Everett, *Units and Physical Constants*.)

Denoting by g the acceleration (in c.g.s. *spouds*) of the attraction of the Earth, then

$$g = CE/R^2, \text{ or } CE = gR^2;$$

so that we can determine E when C is known, and *vice versa*; and till this is done, Newton's *Principia* is merely Kinematics.

With the above value of C , and $g = 981$, $R = 10^9 \div \frac{1}{2}\pi$,

$$E = gR^2/C = 10^{27} \times 6.12 \text{ grammes};$$

giving a mean density $\rho = E/(\frac{4}{3}\pi R^3) = 5.67$.

Denoting by Π the Sun's parallax ($8''.76$), and by T the number of seconds in one year, then the attraction between the Sun and the Earth is

$$F = 4\pi^2ER \operatorname{cosec} \Pi / T^2 = 10^{27} \times 3.65 \text{ dynes},$$

while $S = Fa^2/CE = 10^{33} \times 1.2$ grammes,

with the above numerical values.

180. *Elliptic Planetary Motion.*

Considering an elliptic orbit AP (fig. 55) in which $c \cosh u = OA = a$, $c \sinh u = OB = b$, $c \sinh^2 u / \cosh u = l$, the semi-latus rectum, and $\operatorname{sech} u = e$; then the sector

$$ASP = \text{sector } AOP - \text{triangle } OSP = \frac{1}{2}ab(v - e \sin v).$$

If the ellipse is described in period T round the Sun at S , with mean angular velocity $n = 2\pi/T$, and if the time to P from perihelion at A is denoted by t , then by Kepler's Second Law,

$$\frac{t}{T} = \frac{\frac{1}{2}ab(v - e \sin v)}{\pi ab},$$

or

$$nt = v - e \sin v;$$

and now nt is called the *mean anomaly*, being the true anomaly of a planet moving uniformly in a circular orbit in the same period T .

Expressed in terms of the true anomaly θ from perihelion A ,

$$\cos v = \frac{e + \cos \theta}{1 + e \cos \theta}, \quad \sin v = \frac{\sqrt{(1 - e^2)} \sin \theta}{1 + e \cos \theta},$$

and
$$nt = \sin^{-1} \frac{\sqrt{(1 - e^2)} \sin \theta}{1 + e \cos \theta} - \frac{e \sqrt{(1 - e^2)} \sin \theta}{1 + e \cos \theta}.$$

Reckoned from aphelion A with the Sun at S' ,

$$\begin{aligned} nt' &= v + e \sin v \\ &= \sin^{-1} \frac{\sqrt{(1 - e^2)} \sin \theta'}{1 - e \cos \theta'} + \frac{e \sqrt{(1 - e^2)} \sin \theta'}{1 - e \cos \theta'}. \end{aligned}$$

For completeness we may consider the hyperbolic branch KP , of excentricity e' , described round the Sun at S , and now the mean anomaly

$$\begin{aligned} n't &= e' \sinh u - u \\ &= \frac{e' \sqrt{(e'^2 - 1)} \sin \theta}{1 + e' \cos \theta} - \sinh^{-1} \frac{\sqrt{(e'^2 - 1)} \sin \theta}{1 + e' \cos \theta}, \end{aligned}$$

with

$$n'a'^3 = C(S + E),$$

α' denoting the semi-transverse axis OK , and the true anomaly θ being now KSP , reckoned from perihelion K .

With the Sun at S' , the hyperbolic branch KP will be described under a repulsion from S' , and

$$\begin{aligned} n't' &= e' \sinh u + u \\ &= \frac{e' \sqrt{(e'^2 - 1)} \sin \theta'}{e' \cos \theta' - 1} + \sinh^{-1} \frac{\sqrt{(e'^2 - 1)} \sin \theta'}{e' \cos \theta' - 1}. \end{aligned}$$

181. Rectilinear Motion under Varying Gravity.

By making $e = 1$ or $u = 0$, the ellipse becomes a straight line, and we obtain the solution for rectilinear motion under attraction to a fixed centre, varying inversely as the square of the distance; for instance, the motion of a body shot vertically upwards from the Earth with very great velocity.

Then, if R denotes the radius of the Earth,

$$\frac{d^2x}{dt^2} = -g \frac{R^2}{x^2}, \quad \text{and} \quad \frac{1}{2} \frac{dx^2}{dt^2} = gR^2 \left(\frac{1}{x} - \frac{1}{2a} \right),$$

if the body reaches a distance $2a$ from the centre of the Earth; and if E denotes the weight of the Earth, and C the gravitation constant, $n^2 a^3 = gR^2 = CE$.

Now $x = a(1 + \cos v)$, where $nt = v + \sin v$,
for an attraction; but for a repulsion

$$x = a(\cosh u + 1), \quad \text{where} \quad nt = \sinh u + u.$$

If O denotes the centre of the Earth and the body starting from A falls to P in the time t , and if PQ is drawn at right angles to OA to meet the circle described on OA as diameter in Q ; then the angle $AOQ = \frac{1}{2}v$, and the area $AOQ = \frac{1}{2}a^2(v + \sin v) = \frac{1}{2}a^2nt$; so that the time from A to P is proportional to the area AOQ , and the point Q moves freely on the circle AQ , under an attraction to O .

To shoot a body weighing W lb. vertically upwards away from the Earth to a distance $2a$ feet from the centre, say the distance of the Moon, requires energy, in ft. lb.,

$$\frac{1}{2} Wv^2/g = WR^2\left(\frac{1}{R} - \frac{1}{2a}\right) = WR\left(1 - \frac{R}{2a}\right);$$

and with the Moon's parallax of $57'$, $2a/R = \operatorname{cosec} 57' = 60$, about; while R , the radius of the Earth in feet (§ 99),

$$= 90 \times 60 \times 6080 \div \frac{1}{2}\pi.$$

If we could shoot a body up to a height R at the North Pole, then fired horizontally the body would, in the absence of resistance, skim the surface of the Earth like a grazing satellite, in a period $2\pi/n = 2\pi\sqrt{(R/g)}$ seconds; and $n^2 = g/R = CE/R^3 = \frac{4}{3}\pi\rho C$.

Assuming that gravity inside the Earth varies as the distance from the centre, as would be the case if the Earth were homogeneous, $2\pi\sqrt{(R/g)}$ sec. is also the period of oscillation in a diametral tunnel from pole to pole.

The period of the Moon being about 27 days, the period of a satellite about one-ninth the distance of the Moon would be, by Kepler's Third Law, about one day, and the satellite would appear stationary in the sky; and this would make the period of a grazing satellite $27 \div (60)^{\frac{3}{2}}$ or about one 17th of a day.

In the plane of the equator such a satellite would fly past a point 16 or 18 times a day, every 90 or 80 minutes, according as it moved eastward or westward; and the velocity over the ground would be $7\frac{1}{2}\frac{1}{7}$ or $8\frac{1}{3}$ kilometres per second.

If the Earth was made to rotate 17 times faster, without change of shape, bodies at the equator would be on the point of flying off into space, and elsewhere the plumb line would point to the Pole Star.

182. *Kepler's Problem.*

This is the problem in Astronomy, to express the true anomaly θ and the excentric anomaly v in terms of the mean anomaly nt ; and for this purpose we acquire the series of Lagrange (§ 150) and Fourier (§ 183).

Denoting by m the mean anomaly nt , or more strictly, in astronomical notation, $nt + \epsilon - \varpi$, where ϵ denotes the *epoch*, and ϖ the *longitude of perihelion*, then

$$v = m + e \sin v,$$

which, by Lagrange's Theorem (§ 150), gives in a series proceeding in ascending powers of e ,

$$v = m + \sum \frac{e^p}{p!} \frac{d^{p-1}}{dm^{p-1}} (\sin m)^p;$$

$$\sin v = \sin m + \sum \frac{e^p}{(p+1)!} \frac{d^p}{dm^p} (\sin m)^{p+1}.$$

Similarly $\sin 2v$, $\sin 3v$, ..., $\sin pv$, and $\cos v$ can be expanded in powers of e ; and

$$\frac{r}{a} = 1 - e \cos v = \frac{dm}{dv};$$

$$\frac{a}{r} = \frac{dv}{dm} = 1 + \sum \frac{e^p}{p!} \frac{d^p}{dm^p} (\sin m)^p;$$

while from $\tan \frac{1}{2}\theta = \coth \frac{1}{2}u \tan \frac{1}{2}v$,

we deduce by logarithmic differentiation

$$\frac{d\theta}{\sin \theta} = \frac{dv}{\sin v}, \text{ or } \frac{d\theta}{dv} = \frac{\sinh u}{\cosh u - \cos v} = \frac{1 - c^2}{1 - 2c \cos v + c^2};$$

with $c = e^{-u}$; and, resolved into partial fractions,

$$\frac{d\theta}{dv} = \frac{1}{1 - ce^{iv}} + \frac{1}{1 - ce^{-iv}} - 1$$

$$= 1 + 2 \sum c^p \cos pv,$$

$$\theta = v + 2 \sum \frac{c^p}{p} \sin pv,$$

which can now be expressed in terms of m .

We can rearrange these series so as to proceed according to cosines or sines of multiples of the mean anomaly m , the coefficients being functions of e , and this is more convenient for Kepler's Problem.

A series of this nature is called a *Fourier Series*, and we proceed to show how the coefficients of such a series can be calculated, for any arbitrary single valued function fx , in a series proceeding by cosines or sines of multiples of x/l , when l is any arbitrary quantity.

183. *Fourier's Series.*

Assume that, between the limits $x = \pm l$, the function fx can be expressed by a series of the form

$$fx = \frac{1}{2}A_0 + \Sigma A_p \cos(p\pi x/l) + \Sigma B_p \sin(p\pi x/l),$$

where Σ denotes the sum of the series obtained by giving p all positive integral values from 1 to ∞ ; it is required to determine the A 's and B 's.

Divide the function fx into its *odd* and *even* part (§ 46), and denote them by f_1x and f_2x respectively; thus

$$f_2x = \frac{1}{2}\{f(x) + f(-x)\} = \frac{1}{2}A_0 + \Sigma A_p \cos(p\pi x/l) \dots\dots (i.),$$

$$f_1x = \frac{1}{2}\{f(x) - f(-x)\} = \Sigma B_p \sin(p\pi x/l) \dots\dots (ii.).$$

To determine A_p , change x into v in (i.), multiply both sides by $\cos(p\pi v/l)$, and integrate with respect to v between the limits 0 and l ; then since

$$\begin{aligned} \int_0^l \cos \frac{p\pi v}{l} \cos \frac{q\pi v}{l} dv &= \int_0^l \left\{ \frac{1}{2} \cos(p-q) \frac{\pi v}{l} + \frac{1}{2} \cos(p+q) \frac{\pi v}{l} \right\} dv \\ &= \left[\frac{\sin \{(p-q)\pi v/l\}}{2(p-q)\pi/l} + \frac{\sin \{(p+q)\pi v/l\}}{2(p+q)\pi/l} \right]_0^l = 0; \end{aligned}$$

$$\text{while} \quad \int_0^l \cos^2 \frac{p\pi v}{l} dv = \int_0^l \left(\frac{1}{2} + \frac{1}{2} \cos \frac{2p\pi v}{l} \right) dv = \frac{1}{2}l;$$

$$\text{therefore} \quad \int_0^l f_2v \cos(p\pi v/l) dv = \frac{1}{2}lA_p; \quad \int_0^l f_2v dv = \frac{1}{2}lA_0.$$

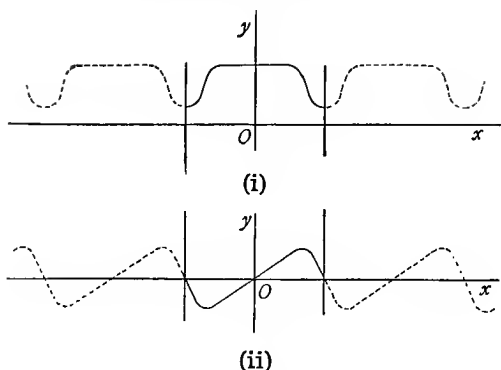


Fig. 56

Similarly, changing x into v in (ii.), multiplying by $\sin(p\pi v/l)$, and integrating with respect to v between 0 and l , we find

$$\int_0^l f_1 v \sin(p\pi v/l) dv = \frac{1}{2} l B_p.$$

Therefore, between the limits $x = \pm l$,

$$\begin{aligned} f x = & \frac{1}{l} \int_0^l f_2 v dv + \frac{2}{l} \sum \cos \frac{p\pi x}{l} \int_0^l f_2 v \cos \frac{p\pi v}{l} dv \\ & + \frac{2}{l} \sum \sin \frac{p\pi x}{l} \int_0^l f_1 v \sin \frac{p\pi v}{l} dv, \end{aligned}$$

and this is called *Fourier's Series*.

By supposing p to take all integral values, positive or negative, we may write Fourier's series,

$$f x = \frac{1}{2l} \sum_{p=-\infty}^{p=\infty} \int_{-l}^l f v \cos \frac{p\pi(x-v)}{l} dv.$$

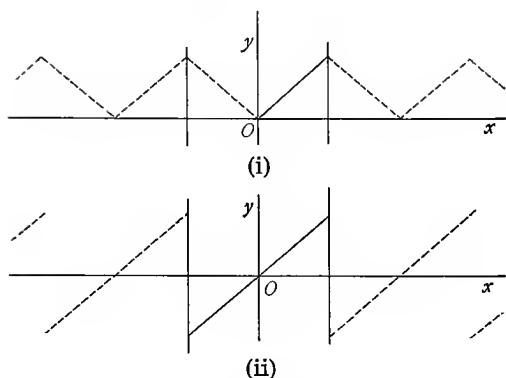


Fig. 57

At the limits $x = \pm l$, the value of the series is $f_2 l$, so that in general there is discontinuity at the limits; and outside these limits Fourier's series represents periodic repetitions of the function fx between the limits.

This is exhibited by drawing the graphs of

(i.) $\frac{1}{2}A_0 + \sum A_p \cos(p\pi x/l)$ and (ii.) $\sum B_p \sin(p\pi x/l)$, as exhibited in fig. 56, (i.) and (ii.)

When the limits between which fx is to be expressed by Fourier's Series are 0 and l , then either series (i.) or (ii.) may be chosen at pleasure, to represent fx , but it is best to choose the series which introduces the least discontinuity at the limits.

For instance, suppose $fx = x$; then between 0 and l ,

$$(i.) \quad x = l - \frac{l}{(\frac{1}{2}\pi)^2} \sum \frac{1}{(2p-1)^2} \cos(2p-1) \frac{\pi x}{l};$$

$$\text{or} \quad (ii.) \quad x = \frac{l}{\frac{1}{2}\pi} \sum \frac{(-1)^{p-1}}{p} \sin \frac{p\pi x}{l};$$

but outside these limits the series (i.) and (ii.) represent the dotted lines in fig. 57, (i.) and (ii.).

The student is recommended to draw the graphs of the first two or three terms in a Fourier series, to see how quickly the series approximates to the given function.

For instance, draw the graphs of

$$\sin x - \frac{1}{2} \sin 2x, \quad \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x, \quad \dots$$

$$\cos x + \frac{1}{3} \cos 3x, \quad \cos x + \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x, \quad \dots$$

Fourier made great use of his series in problems on the Conduction of Heat, where a solution of the partial differential equation $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ is required, giving the

temperature u at any time t ; for if the initial state of temperature U is expressed by the Fourier Series

$$U = \frac{1}{2}A_0 + \Sigma(A_p \cos px + B_p \sin px),$$

then at any subsequent time t the temperature

$$u = \frac{1}{2}A_0 + \Sigma(A_p \cos px + B_p \sin px)e^{-p^2kt}.$$

The geometrical meaning of the coefficients A_p and B_p is explained by Clifford (*Proc. London Math. Soc.*, vol. v.) according to the method employed in Tidal Harmonic Analysis (G. H. Darwin, *Brit. Ass. Report*, 1883), where a pencil attached to a float registers fx , the rise and fall of the tide, on a cylinder turning uniformly by clockwork round a vertical axis.

To analyse the tides of period P , the cylinder is made to revolve in a period P/p , when a closed curve will be traced on the cylinder after p revolutions.

The plane on which the projection of this closed curve has a maximum area A , with attention to the sign of the area (§ 137), will cut the cylinder in an ellipse of area A/p .

Now, if the pencil is made to follow this ellipse as the cylinder revolves in the period P/p , the pencil will have the component harmonic travel

$$A_p \cos(p\pi x/l) + B_p \sin(p\pi x/l).$$

184. *Fourier's Series in Kepler's Problem.*

Given the relation connecting the mean and excentric anomalies,

$$v = m + e \sin v,$$

then
$$\frac{dv}{dm} = \frac{1}{1 - e \cos v} = \frac{1 + e \cos \theta}{1 - e^2} = \frac{a}{r} = 1 + \Sigma A_p \cos pm,$$

suppose, when expressed by a Fourier series; and now

$$\frac{1}{2} A_p = \frac{1}{\pi} \int_0^\pi \frac{dv}{dm} \cos p m dm = \frac{1}{\pi} \int_0^\pi \cos p(v - e \sin v) dv;$$

and this definite integral is a function of pe , called *Bessel's function* of the order p , and denoted by $J_p(pe)$; and it can be verified, as in § 207, that $v = J_m(qr)$ satisfies Bessel's differential equation on p. 185 (§ 88).

Now
$$\frac{a}{r} = \frac{dv}{dm} = 1 + 2 \Sigma J_p(pe) \cos pm;$$

and integrating with respect to m ,

$$v = m + 2 \Sigma \frac{J_p(pe)}{p} \sin pm,$$

giving the excentric anomaly v in terms of the mean anomaly m . Again

$$\sin v \frac{dv}{dm} = \frac{dv}{de} = 2 \Sigma J'_p(pe) \sin pm;$$

and integrating with respect to m ,

$$\cos v = C + 2 \Sigma \frac{J'_p(pe)}{p} \cos pm,$$

where

$$\pi C = \int_0^\pi \cos v dm = \int_0^\pi \cos v (1 - e \cos v) dv = -\frac{1}{2} \pi e, \quad C = -\frac{1}{2} e;$$

and
$$\sin v = \frac{v - m}{e} = 2 \Sigma \frac{J_p(pe)}{pe} \sin pm;$$

whence $x = a \cos v$, $y = b \sin v$, and $r = a(1 - e \cos v)$ can be expressed by a Fourier series in terms of the mean anomaly m .

Examples on Fourier Series.

- (1) Prove that, in a series of cosines, between
- $\pm \frac{1}{2}l$
- ,

$$x^2 = \frac{1}{4}l^2 + \frac{l^2}{(\frac{1}{2}\pi)^3} \sum \frac{(-1)^p}{(2p-1)^3} \cos (2p-1) \frac{\pi x}{l}.$$

Prove that, between 0 and π ,

$$\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots \text{ or } \sum \frac{\sin(2p-1)x}{(2p-1)^3} = \frac{1}{8}\pi^2 x(\pi-x);$$

and give the value of the series outside these limits.

- (2) In a series of cosines,

$$\cosh mx = 2\pi \sinh ml \left\{ \frac{1}{2m^2 l^2} + \sum \frac{(-1)^p}{m^2 l^2 + p^2 \pi^2} \cos \frac{p\pi x}{l} \right\}.$$

- (3) In a series of sines,

$$\sinh mx = 2\pi \sinh ml \sum \frac{(-1)^{p-1}}{m^2 l^2 + p^2 \pi^2} p \sin \frac{p\pi x}{l}.$$

Deduce the expansions of e^{mx} and e^{-mx} .

- (4) Expand $\cos mx$ in a series of cosines, and $\sin mx$ in a series of sines, of multiples of $\pi x/l$.
- (5) Show that the Fourier series which represents the density or temperature in an endless wire must in general contain both sines and cosines.

Determine the Fourier series to represent the temperature of a circular wire, in which the temperatures of the four quadrants in order are 1, 2, 3, 4.

- (6) Prove that the equation

$$y = \frac{1}{2}l + x - \frac{l}{(\frac{1}{2}\pi)^2} \sum \frac{1}{(2p-1)^2} \cos(2p-1) \frac{\pi(x+y)}{l}$$

represents a staircase, of vertical and horizontal steps of length l .

- (7) Prove that the equation

$$\frac{\pi^2}{24} = \sum \frac{(-1)^p}{p^2} \cos \frac{1}{2}p(x+y) \cos \frac{1}{2}p(x-y)$$

represents circles of radius π ; and draw them.

CHAPTER VII.

INTEGRATION IN GENERAL.

185. In the preceding chapters a sketch of Integration has been given, and then a number of applications, intended to show the practical use of the method; and now it is proposed to resume the consideration of Integration from a more general and systematic standpoint, so that the student may more readily perceive and write down the required result.

The process of Integration is necessarily of a tentative nature, depending on a previous knowledge of Differentiation; and in general the most convenient order of the mental operations employed in the integration of a given function will be found to be:

- (i.) to guess the function required for the integration;
- (ii.) to assign the argument of this function;
- (iii.) to write down the proper constant or numerical factors of the integrated function.

Of these three operations, the first is of the most fundamental importance, depending as it does on the principles of the Calculus, but it is the second operation which presents the greatest practical difficulty, while the third only requires verification by a mental differentiation.

186. *General Integration of Algebraical Functions.*

The most general algebraical function of x which is capable of integration by means of the preceding stock of functions can be written

$$\frac{S+T\sqrt{R}}{U+V\sqrt{R}},$$

where S, T, U, V are rational integral algebraical functions of x , and R is a linear or quadratic function of x , and therefore of the form $ax^2+2bx+c$.

If R is of the third or fourth degree in x , *elliptic functions* are in general required for the integration; and if R is of the fifth or higher degree, *hyperelliptic functions* are required.

We first rationalize the denominator, when

$$\frac{S+T\sqrt{R}}{U+V\sqrt{R}} = \frac{(S+T\sqrt{R})(U-V\sqrt{R})}{U^2-V^2R} = \frac{M}{D} + \frac{N}{D} \frac{1}{\sqrt{R}},$$

where $D=U^2-V^2R$, $M=SU-TVR$, $N=(TU-SV)R$;

and D, M, N are thus rational integral functions of x .

To integrate the rational function M/D , this function is split up into its quotient and partial fractions, in the manner explained in treatises on Algebra; and then the integration of each term is in general easily effected.

To integrate the irrational part, $N/D\sqrt{R}$, we may resolve the rational function N/D into its partial fractions, and integrate each term by appropriate substitutions.

187. *Integration of Rational Algebraical Functions.*

To integrate any rational function M/D or fx/Fx , the numerator and denominator of which are rational integral algebraical functions of x (§ 15), and therefore of the form

$$\frac{fx}{Fx} = \frac{ax^m + bx^{m-1} + cx^{m-2} + \dots}{Ax^n + Bx^{n-2} + Cx^{n-3} + \dots}$$

where m and n are positive integers, the function is first resolved into its *partial fractions* by the ordinary rules of Algebra (Smith, *Algebra*, chap. xxiii.; Hall and Knight, *Higher Algebra*, chap. xxiii.); if the degree m of the numerator is equal to or greater than the degree n of the denominator, the *quotient* must be first obtained by division.

Thus to integrate $\frac{x^3}{x^2-3x+2}$, we must suppose it resolved into the form

$$x+3 + \frac{A}{x-1} + \frac{B}{x-2} \equiv \frac{x^3}{x^2-3x+2},$$

$x+3$ being the quotient and $A/(x-1)$, $B/(x-2)$ the partial fractions of the remainder.

To determine the numerator A , multiply both sides of the identity by its denominator $x-1$; then

$$(x-1)(x+3) + A + B \frac{x-1}{x-2} \equiv \frac{x^3}{x-2};$$

and now put $x-1=0$, in the identity: then $A = -1$.

Similarly to determine B , multiply both sides by its denominator $x-2$, and then put this denominator $x-2=0$; this gives $B=8$.

$$\begin{aligned} \text{Now } \int \frac{x^3 dx}{x^2-3x+2} &= \int \left(x+3 - \frac{1}{x-1} + \frac{8}{x-2} \right) dx \\ &= \frac{1}{2}x^2 + 3x - \log(x-1) + 8\log(x-2), \end{aligned}$$

and the integration is effected.

Every rational integral algebraical function of x , such as Fx , can be resolved into real linear factors (factors of the first degree in x), or real quadratic factors (factors of the second degree); thus, for example,

$$x^2 - 1 = (x - 1)(x + 1);$$

$x^2 + 1$ is not decomposable into real linear factors;

$$x^3 - 1 = (x - 1)(x^2 + x + 1);$$

$$x^3 + 1 = (x + 1)(x^2 - x + 1);$$

$$x^4 - 1 = (x - 1)(x + 1)(x^2 + 1);$$

$$x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1);$$

$$x^6 - 1 = (x - 1)(x + 1)(x^2 - x + 1)(x^2 + x + 1);$$

$$x^6 + 1 = (x^2 + 1)(x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1);$$

$$x^8 - 1 = (x - 1)(x + 1)(x^2 + 1)(x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1);$$

$$x^8 + 1 = \{x^2 - \sqrt{(2 + \sqrt{2})x + 1}\} \{x^2 + \sqrt{(2 + \sqrt{2})x + 1}\} \times \\ \{x^2 - \sqrt{(2 - \sqrt{2})x + 1}\} \{x^2 + \sqrt{(2 - \sqrt{2})x + 1}\};$$

and so on.

Corresponding to a quadratic factor in the denominator Fx we must assume a partial fraction with a numerator of the form $Hx + K$; thus, as resolved into partial fractions, we must put $\frac{1}{x^3 - 1} = \frac{A}{x - 1} + \frac{Hx + K}{x^2 + x + 1}$.

The numerator A is then determined as before by multiplying both sides by $x - 1$ and then putting $x - 1 = 0$; thus $A = \frac{1}{3}$; and now, by transposition,

$$\frac{Hx + K}{x^2 + x + 1} = \frac{1}{x^3 - 1} - \frac{1}{3} \frac{1}{x - 1} = \frac{-x - 2}{3(x^2 + x + 1)}$$

(so that $H = -\frac{1}{3}$, $K = -\frac{2}{3}$).

$$\begin{aligned} \text{Now } \int \frac{dx}{x^3 - 1} &= \int \left(\frac{1}{3} \frac{1}{x - 1} - \frac{1}{3} \frac{x + 2}{x^2 + x + 1} \right) dx \\ &= \frac{1}{3} \log(x - 1) - \frac{1}{6} \int \frac{2x + 1}{x^2 + x + 1} dx - \frac{1}{2} \int \frac{dx}{x^2 + x + 1} \\ &= \frac{1}{3} \log(x - 1) - \frac{1}{6} \log(x^2 + x + 1) - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}} \\ &= \frac{1}{6} \log \frac{(x - 1)^3}{x^3 - 1} - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}}. \end{aligned}$$

Examples.—Resolve into partial fractions and integrate

$$(1) \frac{1, x, x^2, x^3}{(x-1)(x-2)}, \frac{1, x, x^2, x^3}{(x-a)(x-b)}, \frac{1, x, x^2, \dots}{(x-1)(x-2)(x-3)},$$

$$\frac{1, x, x^2, \dots}{(x-a)(x-b)(x-c)}, \frac{1, x, x^2, \dots}{(x^2+a^2)(x^2+b^2)}.$$

$$(2) \frac{1, x, x^2, x^3, \dots}{x^2-1, x^2+1, x^3-1, x^3+1}.$$

$$(3) \frac{Ax+B}{(mx+n)^r} \text{ (substitute } mx+n=y),$$

$$\frac{x^p}{(x-a)^m(x-b)^n} \text{ (substitute } \frac{x-a}{x-b}=y),$$

$$\frac{1}{x(ax^m+c)^n} \text{ (substitute } ax^m+c=1/y),$$

$$\frac{1}{(x+1)(x^2-1)}, \frac{1, x, x^2}{(x+1)^2(x^2+1)}.$$

$$(4) \frac{x^2-a^2}{x^4+a^2x^2+a^4}, \frac{1}{x(1+13x^2+36x^4)}.$$

188. Generally, if $x-p$ denotes a real linear factor of the denominator Fx , so that

$$Fx = (x-p)\phi x,$$

we put

$$\frac{fx}{Fx} = \frac{A}{x-p} + \frac{R}{\phi x},$$

and now

$$A = \frac{fx - R(x-p)}{\phi x} = \frac{fp}{\phi p},$$

on putting

$$x-p=0.$$

But

$$(x-p)\phi x = Fx,$$

$$\phi x + (x-p)\phi'x = F'x;$$

so that

$$\phi p = F'p; \text{ and } A = fp/F'p.$$

Now the integral of the corresponding partial fraction $A/(x-p)$ is $A \log(x-p)$.

If $(x-a)^2 + \beta^2$ denotes a quadratic factor of Fx , splitting up into the conjugate imaginary linear factors

$$x-a-i\beta \text{ and } x-a+i\beta,$$

we take two corresponding partial fractions

$$\frac{L+iM}{x-a-i\beta} + \frac{L-iM}{x-a+i\beta},$$

coalescing into the real partial fraction

$$\frac{2L(x-a)-2M\beta}{(x-a)^2+\beta^2},$$

the integral of which is

$$L \log\{(x-a)^2+\beta^2\} - 2M \tan^{-1}\{(x-a)/\beta\}.$$

We generally begin by assuming

$$\frac{fx}{Fx} = \frac{Hx+K}{(x-a)^2+\beta^2} + \frac{S}{\psi x},$$

where

$$Fx = \{(x-a)^2 + \beta^2\} \psi x;$$

and

$$Hx+K = \frac{fx - S\{(x-a)^2 + \beta^2\}}{\psi x} = \frac{fx}{\psi x},$$

on putting $(x-a)^2 + \beta^2 = 0$.

This gives imaginary values of x ; but we may avoid the use of imaginaries by continually writing $2ax - a^2 - \beta^2$ for x^2 , when finally fx and $(Hx+K)\psi x$ are each reduced to the form $Bx+C$ and $B'x+C'$; and thus $B=B'$, $C=C'$, whence H and K are determined.

189. When Fx has a factor $(x-q)^r$, a linear factor $x-q$ repeated r times, so that $Fx = (x-q)^r \chi x$, we must assume r corresponding partial functions of the form

$$\frac{B_r}{(x-q)^r} + \frac{B_{r-1}}{(x-q)^{r-1}} + \dots + \frac{B_1}{x-q} = \frac{fx}{Fx} - \frac{T}{\chi x};$$

and then to determine the B 's put $x-q=y$, or $x=q+y$; so that multiplying by y^r ,

$$B_r + B_{r-1}y + \dots + B_1y^{r-1} = \{f(q+y) - Ty^r\}/\chi(q+y);$$

$B_r, B_{r-1}, \dots B_1$ are thus the coefficients of $y^0, y^1, \dots y^{r-1}$

in the algebraical expansion of $f(q+y)/\chi(q+y)$ in ascending powers of y ; or, in other words,

$$B_r + B_{r-1}y + \dots + B_1y^{r-1}$$

is the quotient and Ty^r is the remainder, when $f(q+y)$ is divided algebraically by $\chi(q+y)$ in ascending powers of y .

Now the integral of a partial fraction $B_s(x-q)^{-s}$ will be $\frac{-B_s}{(s-1)(x-q)^{s-1}}$; and of $\frac{B_1}{x-q}$ will be $B_1 \log(x-q)$.

When Fx has the repeated factor $\{(x-\alpha)^2 + \beta^2\}^r$, we can proceed in the same way, and assume r corresponding partial fractions of the form

$$(H_s x + K_s) / \{(x-\alpha)^2 + \beta^2\}^s;$$

but in this case it is generally preferable to employ the conjugate imaginary factors; or else to employ the substitution $x-\alpha = \beta \tan \theta$ or $\beta \sinh u$.

190. Consider $\int \frac{x^{m-1} dx}{x^n \pm 1}$, where m and n are positive

integers, required in the vertical motion of a body when the resistance varies as the n^{th} power of the velocity.

If $x-\alpha$ is a factor of x^n-1 , then

$$\alpha = 1, \text{ or } \cos(2r\pi/n) + i \sin(2r\pi/n) = \exp(2ir\pi/n),$$

where r has the values $\pm 1, \pm 2, \dots$

For then, by De Moivre's Theorem (§ 111),

$$\alpha^n = \cos 2r\pi + i \sin 2r\pi = 1.$$

$$\text{Put } \frac{x^{m-1}}{x^n-1} = \frac{A}{x-1} + \sum \frac{A_r}{x-\alpha};$$

here $fx = x^{m-1}$, $Fx = x^n - 1$, $F'x = nx^{n-1}$;

$$\begin{aligned} \text{so that } A_r &= \frac{fa}{F'a} = \frac{a^{m-1}}{na^{n-1}} = \frac{a^m}{na^n} = \frac{a^m}{n} \text{ (since } \alpha^n = 1) \\ &= \frac{1}{n} \left(\cos \frac{2mr\pi}{n} + i \sin \frac{2mr\pi}{n} \right) = \frac{1}{n} \exp \frac{2imr\pi}{n}, \end{aligned}$$

by De Moivre's Theorem; also $A = 1/n$.

Therefore
$$\int \frac{x^{m-1} dx}{x^n - 1} = \frac{1}{n} \log(x-1) + \frac{1}{n} \sum \left(\cos \frac{2mr\pi}{n} + i \sin \frac{2mr\pi}{n} \right) \log \left(x - \cos \frac{2r\pi}{n} - i \sin \frac{2r\pi}{n} \right).$$

To express this result in a real form, the partial fractions with numerator A_r and A_{-r} , corresponding to conjugate imaginary values of α , must be combined into a real form as follows—

$$\begin{aligned} & \frac{A_r}{x - \cos(2r\pi/n) - i \sin(2r\pi/n)} + \frac{A_{-r}}{x - \cos(2r\pi/n) + i \sin(2r\pi/n)} \\ &= \frac{1}{n} \cos \frac{2mr\pi}{n} \frac{2x - 2 \cos(2r\pi/n)}{x^2 - 2x \cos(2r\pi/n) + 1} \\ & \quad - \frac{2}{n} \sin \frac{2mr\pi}{n} \frac{\sin(2r\pi/n)}{x^2 - 2x \cos(2r\pi/n) + 1}; \end{aligned}$$

and the corresponding integrals are

$$\begin{aligned} & \frac{1}{n} \cos \frac{2mr\pi}{n} \log(x^2 - 2x \cos \frac{2r\pi}{n} + 1). \\ & - \frac{2}{n} \sin \frac{2mr\pi}{n} \tan^{-1} \left\{ \left(x - \cos \frac{2r\pi}{n} \right) / \sin \frac{2r\pi}{n} \right\}. \end{aligned}$$

When the denominator is $x^n + 1$, the result is of the same form, with $2r-1$ written for $2r$; and from the above we infer that the typical quadratic factors of $x^n - 1$ and $x^n + 1$ are

$$x^2 - 2x \cos(2r\pi/n) + 1, \text{ and } x^2 - 2x \cos\{(2r-1)\pi/n\} + 1.$$

If m is changed into $n-m$, as for instance by the substitution $x=1/y$, then $\cos(2mr\pi/n)$ is unchanged, but $\sin(2mr\pi/n)$ changes in sign, so that (Euler)

$$\int \frac{x^{m-1} \pm x^{n-m-1}}{x^n \pm 1} dx$$

is expressed entirely, either by logarithms, or inverse circular functions.

Degenerate cases occur for $n=m$, or $2m$.

When n is even, $=2p$, and m is odd, $=2q+1$, then, by a rearrangement of terms,

$$\begin{aligned} & \int \frac{x^{m-1} dx}{x^n + 1} = \int \frac{x^{2q} dx}{x^{2p} + 1} \\ &= \frac{1}{2p} \sum_{r=1}^{r=n} \cos \frac{(2r-1)(2q+1)\pi}{2p} \tanh^{-1} \frac{2x \cos(2r-1)\pi/2p}{x^2 + 1} \\ & - \frac{1}{2p} \sum \sin \frac{(2r-1)(2q+1)\pi}{2p} \tan^{-1} \frac{2x \sin(2r-1)\pi/2p}{x^2 - 1}. \end{aligned}$$

Taken between the limits 0 and ∞ , the part depending on the hyperbolic functions vanishes; and the result is

$$\int_0^{\infty} \frac{x^{2q} dx}{x^{2p} + 1} = \frac{\pi}{2p} \sum_{r=1}^{r=p} \frac{\sin \frac{(2r-1)q\pi}{p}}{p} = \frac{\pi}{2p} \operatorname{cosec} \frac{2q+1}{2p} \pi.$$

We may put $x^{2p} = v^n$, and $(2q+1)/2p = m/n$, where m and n are any integers; and now, with the restriction

that $m < n$,

$$\int_0^{\infty} \frac{v^{m-1} dv}{v^n + 1} = \frac{\pi}{n} \operatorname{cosec} \frac{m\pi}{n}.$$

Thus, for instance, the area of the loop of the curve

$$x^n + y^n = a^2 x^{n-m-1} y^{m-1}$$

on putting $y = xv$ (§ 63), is given by

$$\frac{1}{2} \int_0^{\infty} x^2 dv = \frac{1}{2} a^2 \int_0^{\infty} \frac{v^{m-1} dv}{1 + v^n} = \frac{\pi a^2}{2n} \operatorname{cosec} \frac{m\pi}{n}.$$

Putting $v^n = t$, then with the restriction that $m < 1$,

$$\int_0^{\infty} \frac{t^{m-1} dy}{1+t} = \pi \operatorname{cosec} m\pi.$$

To determine $\int_0^{\frac{1}{2}\pi} (\tan \theta)^{q/p} d\theta$, put $\tan \theta = x^p$; then the integral becomes

$$\int_0^{\infty} \frac{p x^{p+q-1} dx}{1+x^{2p}} = \frac{1}{2} \pi \operatorname{cosec} \frac{p+q}{2p} \pi.$$

Similarly for

$$\int_0^{\infty} (\tanh u)^{q/p} du.$$

Examples.—Resolve into partial fractions and integrate

$$(1) \frac{1, x, x^2, x^3 \dots}{x^n - 1 \text{ or } x^n + 1}, \text{ for } n = 4, 6, 8.$$

$$(2) \frac{(1+x^2)^2}{(1-x^2)^3}, \frac{(1+x^2)^4}{(1-x^2)^5}, \dots, \frac{(1-x^2)^2}{(1+x^2)^3}, \frac{(1-x^2)^4}{(1+x^2)^5}, \dots$$

(the integrations required when we integrate $(\sec \theta)^3$, $(\sec \theta)^5$, ... or $(\cosh u)^2$, $(\cosh u)^4$, ... or $\cos^2 \theta$, $\cos^4 \theta$; ... or $(\operatorname{sech} u)^3$, $(\operatorname{sech} u)^5$, ... by means of the substitution $\tan \frac{1}{2} \theta$ or $\tanh \frac{1}{2} u = x$.)

191. *Integration of an Irrational Algebraical Function.*

In integrating the irrational part $N/D\sqrt{R}$ we may suppose the rational function N/D resolved into its quotient and partial fractions; and now we shall consider the integration of the simplest elements,

$$\frac{1}{\sqrt{R}}, \frac{1}{(x-p)\sqrt{R}}, \text{ and } \frac{Hx+K}{(Ax^2+2Bx+C)\sqrt{R}},$$

corresponding to the constant term in the quotient of N/D , and to the partial fractions corresponding with the linear and quadratic factors $x-p$ and $Ax^2+2Bx+C$ of the denominator D ; and afterwards investigate *formulas of reduction* for

$$\frac{x^n}{\sqrt{R}}, \frac{1}{(x-q)^r\sqrt{R}}, \text{ and } \frac{Hx+K}{(Ax^2+2Bx+C)^s\sqrt{R}}.$$

By formulas (i), (w), (x) (§ 39),

$$(i.) \int_0^x \frac{dx}{\sqrt{(m^2-x^2)}} = \sin^{-1} \frac{x}{m} = \cos^{-1} \sqrt{\left(1 - \frac{x^2}{m^2}\right)};$$

$$(ii.) \int_0^x \frac{dx}{\sqrt{(m^2+x^2)}} = \sinh^{-1} \frac{x}{m} = \cosh^{-1} \sqrt{\left(1 + \frac{x^2}{m^2}\right)};$$

$$(iii.) \int_m^x \frac{dx}{\sqrt{(x^2-m^2)}} = \cosh^{-1} \frac{x}{m} = \sinh^{-1} \sqrt{\left(\frac{x^2}{m^2} - 1\right)};$$

in which it will be noticed that the analogy with the first integral breaks down when only the logarithm

$$\log\left\{\sqrt{1+\frac{x^2}{m^2}}+\frac{x}{m}\right\} \text{ or } \log\left\{\frac{x}{m}+\sqrt{\left(\frac{x^2}{m^2}-1\right)}\right\}$$

is employed to express the second and third integral.

The quadratic $R=ax^2+2bx+c$ can always be expressed as the sum or difference of two squares in one of the three forms m^2-x^2 , m^2+x^2 , or x^2-m^2 ; and then the integral $\int dx/\sqrt{R}$ has the corresponding form (i.), (ii.), or (iii.).

But substituting $R=y^2$, we find

$$ax+b=\sqrt{(ay^2+b^2-ac)};$$

so that
$$\int \frac{dx}{\sqrt{R}} = \int \frac{dy}{\sqrt{(ay^2+b^2-ac)}};$$

and now, as in ex. 13, p. 77,

(i.) when a is negative, but b^2-ac positive,

$$\begin{aligned} \int \frac{dx}{\sqrt{R}} &= \frac{1}{\sqrt{(-a)}} \sin^{-1} \frac{\sqrt{(-a)}y}{\sqrt{(b^2-ac)}} \\ &= \frac{1}{\sqrt{(-a)}} \sin^{-1} \frac{\sqrt{(-a)}\sqrt{R}}{\sqrt{(b^2-ac)}} = \frac{1}{\sqrt{(-a)}} \cos^{-1} \frac{ax+b}{\sqrt{(b^2-ac)}}; \end{aligned}$$

(ii.) when a is positive, and b^2-ac positive,

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{(a)}} \sinh^{-1} \frac{\sqrt{(a)}\sqrt{R}}{\sqrt{(b^2-ac)}} = \frac{1}{\sqrt{(a)}} \cosh^{-1} \frac{ax+b}{\sqrt{(b^2-ac)}};$$

(iii.) when a is positive, and b^2-ac negative,

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{(a)}} \cosh^{-1} \frac{\sqrt{(a)}\sqrt{R}}{\sqrt{(ac-b^2)}} = \frac{1}{\sqrt{(a)}} \sinh^{-1} \frac{ax+b}{\sqrt{(ac-b^2)}}.$$

We cannot have both a and b^2-ac negative, because R would then be negative, and \sqrt{R} imaginary, for all real values of x .

When $a=0$ the integral assumes an indeterminate form (§ 120), and reduces to an algebraical function

$$\sqrt{R}/b, \text{ or } \sqrt{(2bx+c)}/b.$$

When $b^2 - ac$ is positive, R breaks up into real linear factors, say $x - \alpha$ and $x - \beta$; and now

(1) when a is negative, we may without loss of generality put $a = -1$; and

$$\begin{aligned}\int \frac{dx}{\sqrt{R}} &= \int \frac{dx}{\sqrt{\{(a-x)(x-\beta)\}}} = \sin^{-1} \frac{\sqrt{(a-x)(x-\beta)}}{\frac{1}{2}(a-\beta)} \\ &= 2 \sin^{-1} \sqrt{\frac{x-\beta}{a-\beta}} = 2 \cos^{-1} \sqrt{\frac{a-x}{a-\beta}} = 2 \tan^{-1} \sqrt{\frac{x-\beta}{a-x}}.\end{aligned}$$

(2) when a is positive, we may put $a = 1$, and

$$\begin{aligned}\int \frac{dx}{\sqrt{R}} &= \int \frac{dx}{\sqrt{(x-a)(x-\beta)}} = \sinh^{-1} \frac{\sqrt{(x-a)(x-\beta)}}{\frac{1}{2}(a-\beta)} \\ &= 2 \sinh^{-1} \sqrt{\frac{x-a}{a-\beta}} = 2 \cosh^{-1} \sqrt{\frac{x-\beta}{a-\beta}} = 2 \tanh^{-1} \sqrt{\frac{x-a}{x-\beta}} \\ &= 2 \log \frac{\sqrt{(x-a)} + \sqrt{(x-\beta)}}{\sqrt{(a-\beta)}}.\end{aligned}$$

We may write, by analogy,

$$\begin{aligned}\int \frac{dx}{R} &= \frac{1}{\sqrt{(ac-b^2)}} \sec^{-1} \frac{\sqrt{a}\sqrt{R}}{\sqrt{(ac-b^2)}}, \\ \text{or} \quad & -\frac{1}{\sqrt{(b^2-ac)}} \operatorname{sech}^{-1} \frac{\sqrt{(-a)}\sqrt{R}}{\sqrt{(b^2-ac)}}, \\ \text{or} \quad & -\frac{1}{\sqrt{(b^2-ac)}} \operatorname{cosech}^{-1} \frac{\sqrt{a}\sqrt{R}}{\sqrt{(b^2-ac)}},\end{aligned}$$

thus exhibiting the result as a function of R .

For instance

$$\begin{aligned}\int_x^\infty \frac{dx}{x^2 - 2ax \cos \alpha + a^2} &= \frac{1}{a \sin \alpha} \operatorname{cosec}^{-1} \frac{\sqrt{(x^2 - 2ax \cos \alpha + a^2)}}{a \sin \alpha}; \\ \int_x^\infty \frac{dx}{x^2 - 2ax \cosh \beta + a^2} &= \frac{1}{a \sinh \beta} \operatorname{cosech}^{-1} \frac{\sqrt{(x^2 - 2ax \cosh \beta + a^2)}}{a \sinh \beta}; \\ \int_x^\infty \frac{dx}{x^2 - 2ax \sinh \gamma - a^2} &= \frac{1}{a \cosh \gamma} \operatorname{cosech}^{-1} \frac{\sqrt{(x^2 - 2ax \sinh \gamma - a^2)}}{a \cosh \gamma}.\end{aligned}$$

192. Next to determine $\int \frac{dx}{(x-p)\sqrt{R}}$, we may substitute $x-p=1/y$, when we obtain an integral of the same form as $\int dx/\sqrt{R}$; but it is more direct to substitute $y=\sqrt{R}/(x-p)$; and now

$$y^2 + \lambda = \frac{R + \lambda(x-p)^2}{(x-p)^2} = \frac{(a+\lambda)x^2 + 2(b-p\lambda)x + c + \lambda p^2}{(x-p)^2};$$

a perfect square when $\lambda = \frac{b^2 - ac}{ap^2 + 2bp + c}$; so that

$$(ap^2 + 2bp + c)y^2 + b^2 - ac = \left\{ \frac{(ap+b)x + bp + c}{x-p} \right\}^2;$$

and taking the negative sign with the radical

$$\frac{(ap+b)x + bp + c}{x-p} = -\sqrt{\{(ap^2 + 2bp + c)y^2 + b^2 - ac\}}$$

$$\begin{aligned} \frac{dx}{(x-p)^2} &= \frac{y dy}{\sqrt{\{(ap^2 + 2bp + c)y^2 + b^2 - ac\}}} \\ \int \frac{dx}{(x-p)\sqrt{R}} &= \int \frac{dx}{(x-p)^2 y} = \int \frac{dy}{\sqrt{\{(ap^2 + 2bp + c)y^2 + b^2 - ac\}}} \\ &= \frac{1}{\sqrt{(-ap^2 - 2bp - c)}} \sin^{-1} \frac{\sqrt{(-ap^2 - 2bp - c)}y}{\sqrt{(b^2 - ac)}} \\ &= \frac{1}{\sqrt{(-ap^2 - 2bp - c)}} \sin^{-1} \frac{\sqrt{(-ap^2 - 2bp - c)}\sqrt{R}}{\sqrt{(b^2 - ac)(x-p)}} \dots (i.); \\ \text{or} &= \frac{1}{\sqrt{(ap^2 + 2bp + c)}} \sinh^{-1} \frac{\sqrt{(ap^2 + 2bp + c)}\sqrt{R}}{\sqrt{(b^2 - ac)(x-p)}} \dots (ii.); \\ \text{or} &= \frac{1}{\sqrt{(ap^2 + 2bp + c)}} \cosh^{-1} \frac{\sqrt{(ap^2 + 2bp + c)}\sqrt{R}}{\sqrt{(ac - b^2)(x-p)}} \dots (iii.); \end{aligned}$$

the real form to be chosen (ex. 18, p. 78).

Here again, when $ap^2 + 2bp + c = 0$, so that $x-p$ is a factor of R , the integral assumes an indeterminate form and the limiting value is the algebraical function

$$\frac{\sqrt{R}}{\sqrt{(b^2 - ac)(x-p)}}.$$

193. We may consider the integral $\int dx/(x-p)\sqrt{R}$ as the degenerate form of the *elliptic integral* $\int dx/\sqrt{X}$, where X is a quartic function of x , as in ex. 39, p. 82, when X splits up into the factors $(x-p)^2$ and R .

Similarly the *canonical elliptic integral* $\int ds/\sqrt{S}$, where, as in ex. 39, p. 82,

$$S = 4s^3 - g_2s - g_3 = 4(s-e_1)(s-e_2)(s-e_3),$$

when resolved into factors, becomes one of the degenerate forms of exs. 14, 15, p. 77, if two of the quantities e_1, e_2, e_3 are equal.

We suppose $e_1 > e_2 > e_3$; and that the middle quantity e_2 is made equal either to the smallest e_3 , or the greatest e_1 ; and then with (i.) $e_2 = e_3$, as in ex. 14, p. 77,

$$\begin{aligned} \int \frac{ds}{\sqrt{S}} &= \int \frac{ds}{2(x-e_3)\sqrt{(x-e_1)}} \\ &= \frac{1}{\sqrt{(e_1-e_3)}} \sin^{-1} \sqrt{\frac{x-e_1}{x-e_3}} = \frac{1}{2\sqrt{(e_1-e_3)}} \sin^{-1} \frac{2\sqrt{(e_1-e_3)(x-e_1)}}{x-e_3}; \end{aligned}$$

$$(ii) \quad e_2 = e_1, \int \frac{ds}{\sqrt{S}}$$

$$= \int \frac{ds}{2(e_1-x)\sqrt{(x-e_3)}} = \frac{1}{\sqrt{(e_1-e_3)}} \sinh^{-1} \sqrt{\frac{x-e_3}{e_1-x}} \quad (x < e_1),$$

$$\text{or} \quad = \frac{1}{\sqrt{(e_1-e_3)}} \cosh^{-1} \sqrt{\frac{x-e_3}{x-e_1}} \quad (x > e_1).$$

Similarly, as in ex. 15, p. 77,

$$\int \frac{ds}{2(e_1-x)\sqrt{(e_3-x)}} = \frac{1}{\sqrt{(e_1-e_3)}} \cos^{-1} \sqrt{\frac{e_3-x}{e_1-x}};$$

$$\int \frac{ds}{2(e_3-x)\sqrt{(e_1-x)}} = \frac{1}{\sqrt{(e_1-e_3)}} \cosh^{-1} \sqrt{\frac{e_1-x}{e_3-x}} \quad (x < e_3),$$

$$\text{or} \quad = \frac{1}{\sqrt{(e_1-e_3)}} \sinh^{-1} \sqrt{\frac{e_1-x}{x-e_3}} \quad (x > e_3).$$

In physical problems we require the integrals

$$A = \int_{\lambda}^{\infty} \frac{d\lambda}{(a^2 + \lambda)^2 (c^2 + \lambda)^{\frac{1}{2}}} \text{ and } C = \int_{\lambda}^{\infty} \frac{d\lambda}{(a^2 + \lambda)(c^2 + \lambda)^{\frac{3}{2}}}.$$

$$\text{Now } \int_{\lambda}^{\infty} \frac{d\lambda}{(a^2 + \lambda)\sqrt{(c^2 + \lambda)}} = \frac{2}{\sqrt{(a^2 - c^2)}} \cos^{-1} \sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}},$$

$$\text{or } = \frac{2}{\sqrt{(c^2 - a^2)}} \cosh^{-1} \sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}};$$

and, differentiating with respect to a^2 or c^2 , we deduce

$$A = -\frac{\sqrt{(c^2 + \lambda)}}{(a^2 - c^2)(a^2 + \lambda)} + \frac{1}{(a^2 - c^2)^{\frac{3}{2}}} \cos^{-1} \sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}},$$

$$\text{or } -\frac{\sqrt{(c^2 + \lambda)}}{(c^2 - a^2)(a^2 + \lambda)} + \frac{1}{(c^2 - a^2)^{\frac{3}{2}}} \cosh^{-1} \sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}};$$

$$C = \frac{2}{(a^2 - c^2)\sqrt{(c^2 + \lambda)}} - \frac{2}{(a^2 - c^2)^{\frac{3}{2}}} \cos^{-1} \sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}},$$

$$\text{or } \frac{2}{(c^2 - a^2)\sqrt{(c^2 + \lambda)}} - \frac{2}{(c^2 - a^2)^{\frac{3}{2}}} \cosh^{-1} \sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}};$$

$$\text{so that } 2A + C = \frac{2}{(a^2 + \lambda)\sqrt{(c^2 + \lambda)}}.$$

194. To determine the integral

$$\int \frac{Hx + K}{Ax^2 + 2Bx + C} \frac{dx}{\sqrt{(ax^2 + 2bx + c)}},$$

where we may suppose A and therefore $Ax^2 + 2Bx + C$ positive for all real values of x , substitute

$$y = (ax^2 + 2bx + c)/(Ax^2 + 2Bx + C).$$

Now, with the notation of ex. 8, p. 146,

$$\frac{dy}{dx} = \frac{2(Ab - aB)(x_1 - x)(x - x_2)}{(Ax^2 + 2Bx + C)^2},$$

$$y_1 - y = \frac{(Ay_1 - a)(x_1 - x)^2}{Ax^2 + 2Bx + C}, \quad y - y_2 = \frac{(a - Ay_2)(x - x_2)^2}{Ax^2 + 2Bx + C};$$

where y_1, y_2 denote the maximum and minimum of y , and x_1, x_2 the corresponding values of x .

We may write $L(x-x_2)+M(x_1-x)$ for $Hx+K$, and

$$V=Ax^2+2Bx+C=P(x_1-x)^2+Q(x-x_2)^2,$$

$$R=ax^2+2bx+c=p(x_1-x)^2+q(x-x_2)^2,$$

as before, on p. 147; and now we find

$$\begin{aligned} & \int \frac{L(x-x_2)+M(x_1-x)}{V\sqrt{R}} dx \\ &= \int \frac{L(x-x_2)+M(x_1-x)}{V^{\frac{3}{2}}\sqrt{y}} \frac{V^2 dy}{2(Ab-aB)(x_1-x)(x-x_2)} \\ &= \frac{1}{2}LL' \int \frac{dy}{\sqrt{\{y(y_1-y)\}}} + \frac{1}{2}MM' \int \frac{dy}{\sqrt{\{y(y-y_2)\}}} \end{aligned}$$

where $L' = \frac{\sqrt{(Ay_1-a)}}{Ab-aB}$, $M' = \frac{\sqrt{(a-Ay_2)}}{Ab-aB}$;

so that the expression for the integral is

$$-LL' \cos^{-1} \sqrt{(y/y_1)} + MM' \cosh^{-1} \sqrt{(y/y_2)},$$

or $LL' \sin^{-1} \sqrt{(y/y_1)} + MM' \sinh^{-1} \sqrt{(-y/y_2)},$

according as y_2 is positive or negative; that is, according as $ac-b^2$ is positive or negative; that is, as R is always positive, or as R can vanish, for real values of x .

When $V=R$, these forms become illusory; and now

$$\int R^{-\frac{3}{2}} dx = (ax+b)/(ac-b^2) \sqrt{R},$$

$$\int (ax+b)R^{-\frac{3}{2}} dx = R^{-\frac{1}{2}};$$

whence $\int (Hx+K)R^{-\frac{3}{2}} dx$ can be determined, as an algebraical function of x ; and similarly $\int (Hx+K)R^{-\frac{5}{2}} dx, \dots$

Again, when $Ab-aB=0$, the results are again illusory; but in this case we can choose a new variable, by changing $x+b/a$ or $x+B/A$ into x , so that we may make b and B vanish; and now the integral

$$\int \frac{Lx+M}{Ax^2+C} \frac{dx}{\sqrt{(ax^2+c)}}$$

consists of two parts, of which the first is a function of x^2 , of the form given in § 193; while, as in ex. 19, p. 79,

$$\int \frac{Mdx}{(Ax^2+C)\sqrt{(ax^2+c)}} = \frac{M}{\sqrt{C}\sqrt{(Ac-aC)}} \cos^{-1} \sqrt{\left(\frac{C}{c} \frac{ax^2+c}{Ax^2+C}\right)},$$

$$\text{or} \quad \frac{M}{\sqrt{C}\sqrt{(aC-Ac)}} \cosh^{-1} \sqrt{\left(\frac{C}{c} \frac{ax^2+c}{Ax^2+C}\right)},$$

$$\text{or} \quad \frac{M}{\sqrt{C}\sqrt{(aC-Ac)}} \sinh^{-1} \sqrt{\left(-\frac{C}{c} \frac{ax^2+c}{Ax^2+C}\right)};$$

reducing, when $aC - Ac = 0$, to $\frac{x}{C\sqrt{(ax^2+c)}}$.

By differentiation of the integral

$$\int \frac{Hx+K}{Ax^2+2Bx+C} \frac{dx}{\sqrt{(ax^2+2bx+c)}}$$

with respect to A , B , or C , we can deduce the results of

$$\int \frac{fx}{(Ax^2+2Bx+C)^n} \frac{dx}{\sqrt{(ax^2+2bx+c)}}.$$

The general linear substitution may be written

$$x' = \frac{e(x_1-x) + f(x-x_2)}{E(x_1-x) + F(x-x_2)},$$

and the form of the integral will then be unchanged; in particular V and R are interchanged if we put

$$E = \sqrt{(Pp)}, F = \sqrt{(Qq)}, e = Ex_1, f = Fx_2.$$

Examples.

(1) Integrate

$$\begin{aligned} & \frac{x-1}{(x^2+x+1)\sqrt{(x^2-x+1)}} \frac{x+1}{(x^2-x+1)\sqrt{(x^2+x+1)}}, \\ & \frac{L(x-1)+M(x-2)}{(3x^2-10x+9)\sqrt{(5x^2-16x+14)}}, \\ & \frac{L(x-1)+M(x-2)}{(5x^2-16x+14)\sqrt{(3x^2-10x+9)}}, \\ & \frac{Hx+K}{(3x^2-10x+9)\sqrt{(x^2-8x+10)}}. \end{aligned}$$

- (2) Prove that the maximum and minimum of
 $(ax^2 + 2hxy + by^2)/(Ax^2 + 2Hxy + By^2)$
 are given by the common conjugate diameters of
 $ax^2 + 2hxy + by^2 = c$, $Ax^2 + 2Hxy + By^2 = C$.

- (3) With the notation of ex. 8, p. 146, prove that

$$ax + b = -p(x_1 - x) + q(x - x_2),$$

$$bx + c = px_1(x_1 - x) - qx_2(x - x_2);$$

$$Ax + B = -P(x_1 - x) + Q(x - x_2),$$

$$Bx + C = Px_1(x_1 - x) - Qx_2(x - x_2);$$

$$\text{and } (B^2 - AC)R^2 + (Ac + aC - 2Bb)RV + (b^2 - ac)V^2 \\ = (Ab - aB)^2(x_1 - x)^2(x - x_2)^2.$$

$$\text{If } x' = -(Bx + C)/(Ax + B), \text{ or } -(bx + c)/(ax + b),$$

$$\frac{R}{V} + \frac{R'}{V'} = \frac{Ac + aC - 2Bb}{AC - B^2}, \text{ or } \frac{V}{R} + \frac{V'}{R'} = \frac{Ac + aC - 2Bb}{ac - b^2}.$$

- (4) Prove that if X denotes the reciprocal quartic

$$ax^4 + 4bx^3 + 6cx^2 + 4bx + a,$$

$$(i.) \int \frac{x^2 - 1}{x} \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{(-a)}} \sin^{-1} \frac{\sqrt{(-a)}\sqrt{X}}{\sqrt{(-D)x}},$$

$$\text{or } \frac{1}{\sqrt{a}} \sinh^{-1} \frac{\sqrt{a}\sqrt{X}}{\sqrt{(-D)x}}, \text{ or } \frac{1}{\sqrt{a}} \cosh^{-1} \frac{\sqrt{a}\sqrt{X}}{\sqrt{(D)x}};$$

$$\text{where } aX = (ax^2 + 2bx + a)^2 + Dx^2, D = 6ac - 4b^2 - 2a^2.$$

$$(ii.) \int \frac{x^2 - 1}{x^2 + 1} \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{(2a - 6c)}} \sin^{-1} \frac{\sqrt{(2a - 6c)}\sqrt{X}}{\sqrt{(-D)(x^2 + 1)}},$$

$$\text{or } \frac{1}{\sqrt{(6c - 2a)}} \sinh^{-1} \frac{\sqrt{(6c - 2a)}\sqrt{X}}{\sqrt{(-D)(x^2 + 1)}},$$

$$\text{or } \frac{1}{\sqrt{(6c - 2a)}} \cosh^{-1} \frac{\sqrt{(6c - 2a)}\sqrt{X}}{\sqrt{(D)(x^2 + 1)}};$$

$$\text{and } (6c - 2a)X = 4\{bx^2 - (a - 3c)x + b\}^2 + D(x^2 + 1)^2.$$

- (5) Determine in a similar form

$$\int \frac{x^2 + 1}{x} \frac{dx}{\sqrt{X}}, \text{ and } \int \frac{x^2 + 1}{x^2 - 1} \frac{dx}{\sqrt{X}},$$

$$\text{where } X' = ax^4 - 4bx^3 + 6cx^2 + 4bx + a.$$

195. *Integration by Rationalization.*

Any integral of the form

$$\int \frac{S + T(ax + b)^{p/r}}{U + V(ax + b)^{q/r}} dx,$$

where S, T, U, V are rational integral functions of x , can be *rationalized*, that is, can be made to depend on the integration of a rational algebraical fraction by the substitution $ax + b = y^r$; for then $adx = ry^{r-1}dy$; and the integral becomes the rational algebraical integral

$$\int \frac{S + Ty^p}{U + Vy^q} \frac{ry^{r-1}}{a} dy.$$

Any integral of the form

$$\int \frac{S + T\sqrt{R}}{U + V\sqrt{R}} dx,$$

where R is the quadratic form $ax^2 + 2bx + c$, can be rationalized,

(i.) when $b^2 - ac$ is positive, and a negative, by the substitution

$$\frac{\sqrt{(-a)}\sqrt{R}}{\sqrt{(b^2 - ac)}} = \frac{2y}{1 + y^2}, \text{ then } \frac{ax + b}{\sqrt{(b^2 - ac)}} = \frac{1 - y^2}{1 + y^2}$$

and

$$\frac{-adx}{\sqrt{(b^2 - ac)}} = \frac{4ydy}{(1 + y^2)^2};$$

(ii.) when $b^2 - ac$ is positive, and a positive, by

$$\frac{\sqrt{(a)}\sqrt{R}}{\sqrt{(b^2 - ac)}} = \frac{2y}{1 - y^2}, \quad \frac{ax + b}{\sqrt{(b^2 - ac)}} = \frac{1 + y^2}{1 - y^2},$$

and

$$\frac{adx}{\sqrt{(b^2 - ac)}} = \frac{4ydy}{(1 - y^2)^2};$$

(iii.) when $b^2 - ac$ is negative, and a positive, by

$$\frac{\sqrt{(a)}\sqrt{R}}{\sqrt{(ac - b^2)}} = \frac{1 + y^2}{1 - y^2}, \quad \frac{ax + b}{\sqrt{(ac - b^2)}} = \frac{2y}{1 - y^2},$$

and

$$\frac{adx}{\sqrt{(ac - b^2)}} = 2 \frac{1 + y^2}{(1 - y^2)^2} dy.$$

In (i.) we may suppose $y = \tan \frac{1}{2}\theta$, and in (ii.) and (iii.) $y = \tanh \frac{1}{2}u$; and then the integration is changed to an integration of a rational function of $\cos \theta$ and $\sin \theta$, or $\cosh u$ and $\sinh u$, with respect to θ or u .

Examples.—Integrate with respect to x

$$\begin{aligned} & \frac{x^3}{\sqrt{(x-a)}}, \frac{x^n}{\sqrt{(x+a)}}, x^n \sqrt{(x-a)}, \frac{1}{(1+x)\sqrt{x}}, \\ & \frac{1}{x^4 \sqrt{(1+x^2)}}, \frac{1}{(x+1)\sqrt{(x+2)}}, \frac{x}{(x+2)\sqrt{(x+1)}}, \\ & \frac{1}{(x+1)\sqrt{(2x+1)}}, \frac{1}{(1-x)\sqrt{(1-x^2)}}, \frac{1}{(1+x^2)\sqrt{(1-x^2)}}, \\ & \frac{1}{\sqrt{(a^2+x^2)}}, \frac{1}{a+x}, \frac{1}{x\sqrt{(x^2+3x+2)}}, \frac{1}{(x+1)\sqrt{(x^2+x+1)}}, \\ & \frac{1}{(x^2+1)^{\frac{3}{2}}}, \frac{1}{(ax^2+2bx+c)^{\frac{3}{2}}}, \frac{bx+c}{(ax^2+2bx+c)^{\frac{3}{2}}}, \\ & \frac{Hx+K}{(ax^2+2bx+c)^{\frac{3}{2}}}, \frac{1}{x^3 \sqrt{(ax^2+c)}}, (1-2x^2)^{\frac{3}{2}}, \\ & \frac{x}{\sqrt{(x^2+a^2 \cdot x^2+b^2)}}, \frac{1}{(x^4-1)^{\frac{1}{4}}}, \sqrt{(1+e^x)}, \\ & (ax^n+b)^{-\frac{n+1}{n}}, \text{ (substitute } a+bx^{-n}=y\text{).} \end{aligned}$$

196. *Integration of Circular and Hyperbolic Functions.*

To integrate powers and products of $\cos x$ and $\sin x$ the most general plan is to convert them into cosines and sines of multiples of x , which are immediately integrable (ex. 9, § 40).

To integrate any odd power of $\cos x$ or $\sin x$, say $(\cos x)^{2n+1}$ or $(\sin x)^{2n+1}$, we write them in the form

$$(1 - \sin^2 x)^n \cos x, \text{ or } (1 - \cos^2 x)^n \sin x,$$

and expand by the Binomial Theorem; then each term is immediately integrable, since by § 40 and (a), p. 84,

$$\int (\sin x)^m \cos x dx = \frac{(\sin x)^{m+1}}{m+1},$$

$$\int (\cos x)^m \sin x dx = -\frac{(\cos x)^{m+1}}{m+1}.$$

A similar method will serve to integrate $(\sin x)^p(\cos x)^q$, where either p or q is an odd integer: also to integrate any powers of $\text{vers } x$.

The same processes apply when $\cos x$ and $\sin x$ are replaced by the hyperbolic functions $\cosh x$ and $\sinh x$.

Examples.—Integrate with respect to x , $\cos mx$, $\sin(mx+n)$, $\sin 2x \cos 3x$, $\cos 3x \cos 5x$, $\sin 3x \sin 5x$, $\sin(mx+n)\cos(px+q)$, $\sin x \sin 2x \sin 3x$, $\sin x \cos x$, $\sin^2 x \cos x$, $(\sin x)^3 \cos x$, $(\sin x)^m \cos x$, $\sin x \cos^2 x$, $\sin x \cos^3 x$, $\sin x (\cos x)^m$, $\sin^2 x$, $\sin^3 x$, $\sin^4 x$, $\cos^2 x$, $\cos^3 x$, $\cos^4 x$, $\cos^2 mx \cos nx$, $\cos^3 mx \cos nx$, ..., also the same functions with $\cosh x$ for $\cos x$, and $\sinh x$ for $\sin x$.

197. The integration of the remaining trigonometrical functions is a little more complicated; thus, by (v), p. 85,

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \log \sin x = \frac{1}{2} \log \text{vers } 2x;$$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\log \cos x = \log \sec x;$$

while $\int \coth x dx = \log \sinh x$, $\int \tanh x dx = \log \cosh x$.

$$\text{Again } \int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x,$$

$$\int \cot^2 x dx = \int (\text{cosec}^2 x - 1) dx = -\cot x - x;$$

with similar results for $\tanh^2 x$ and $\coth^2 x$; while other powers of $\tan x$ are integrated by expressing them as the sum of terms of the form $(\sec x)^p \tan x$ or $(\tan x)^q \sec^2 x$, the integrals of which are $(\sec x)^p/p$ and $(\tan x)^{q+1}/(q+1)$; similarly for powers of $\cot x$, $\tanh x$, or $\coth x$.

To integrate $\sec x$ and $\operatorname{cosec} x$, we may rationalize by means of the substitution $\tan \frac{1}{2}x = y$; and then

$$\cos x = \frac{1-y^2}{1+y^2}, \quad \sin x = \frac{2y}{1+y^2},$$

while $x = 2 \tan^{-1} y, \quad dx = 2dy/(1+y^2).$

$$\begin{aligned} \text{Then } \int \sec x dx &= \int \frac{1+y^2}{1-y^2} \frac{2dy}{1+y^2} = \int \frac{2dy}{1-y^2} \\ &= \int \left(\frac{1}{1+y} + \frac{1}{1-y} \right) dy \\ &= \log(1+y) - \log(1-y) = \log(1+y)/(1-y) \\ &= \log \frac{1+\tan \frac{1}{2}x}{1-\tan \frac{1}{2}x} = \log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x) = \log \sqrt{\frac{1+\sin x}{1-\sin x}} \\ &= \log(\sec x + \tan x). \end{aligned}$$

$$\begin{aligned} \text{Similarly } \int \operatorname{cosec} x dx &= \int \frac{1+y^2}{2y} \frac{2dy}{1+y^2} \\ &= \int dy/y = \log y = \log \tan \tfrac{1}{2}x = \log \sqrt{\frac{1-\cos x}{1+\cos x}} \\ &= \log(\operatorname{cosec} x - \cot x). \end{aligned}$$

Similarly to integrate $\operatorname{sech} x$ or $\operatorname{cosech} x$, we may rationalize by the substitution $\tanh \frac{1}{2}x = y$; and then

$$\begin{aligned} \int \operatorname{sech} x dx &= \int \frac{2dy}{1+y^2} = 2 \tan^{-1} y = 2 \tan^{-1} \tanh \tfrac{1}{2}x, \\ \int \operatorname{cosech} x dx &= \int dy/y = \log y = \log \tanh \tfrac{1}{2}x. \end{aligned}$$

The substitution of y for $\tan \frac{1}{2}\theta$ will apply for any function of the form $f(\cos \theta, \sin \theta)/F(\cos \theta, \sin \theta)$; the function being thereby reduced to the form

$$\frac{\prod(y-b)}{\prod(y-a)} \text{ or } \frac{\prod \sin \frac{1}{2}(\theta-\beta)}{\prod \sin \frac{1}{2}(\theta-\alpha)};$$

and this again by partial fractions (§ 187) to the sum of terms of the form $A/(y-a)$, or

$$A \cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}\alpha \operatorname{cosec} \tfrac{1}{2}(\theta-\alpha) = A \cos^2 \tfrac{1}{2}\alpha \cot \tfrac{1}{2}(\theta-\alpha) - \tfrac{1}{2} A \sin \alpha.$$

(Hermite, *Proc. London Math. Society*, vol. IV.).

198. By the substitution of z for $\sec x$, $\operatorname{cosec} x$, $\operatorname{sech} x$, $\operatorname{cosech} x$, the results can be expressed more directly; thus

$$\text{with } z = \sec x, \quad dx = \frac{dz}{z\sqrt{(z^2-1)}},$$

$$\text{and } \int \sec x dx = \int \frac{dz}{\sqrt{(z^2-1)}} = \cosh^{-1} z, \text{ by (x.) (p. 85)}$$

$$= \cosh^{-1} \sec x = \sinh^{-1} \tan x = \tanh^{-1} \sin x = 2 \tanh^{-1} \tan \frac{1}{2}x.$$

Similarly, with $z = \operatorname{cosec} x$, and as a *corrected* integral,

$$\begin{aligned} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \operatorname{cosec} x dx &= \int_1^{-1} \frac{dz}{\sqrt{(z^2-1)}} = \cosh^{-1} z \\ &= \cosh^{-1} \operatorname{cosec} x = \sinh^{-1} \cot x = \tanh^{-1} \cos x. \end{aligned}$$

Again, with $z = \operatorname{sech} x$, or $\operatorname{cosech} x$,

$$\begin{aligned} \int_0^1 \operatorname{sech} x dx &= \int_1^0 \frac{dz}{\sqrt{(1-z^2)}} = \cos^{-1} z \\ &= \cos^{-1} \operatorname{sech} x = \sin^{-1} \tanh x = \tan^{-1} \sinh x = 2 \tan^{-1} \tanh \frac{1}{2}x. \end{aligned}$$

$$\begin{aligned} \int_0^\infty \operatorname{cosech} x dx &= \int_0^1 \frac{dz}{\sqrt{(1+z^2)}} = \sinh^{-1} z \\ &= \sinh^{-1} \operatorname{cosech} x = \cosh^{-1} \coth x = \log \coth \frac{1}{2}x. \end{aligned}$$

199. To integrate $1/P$, where $P = a + b \cos x + c \sin x$, we can proceed in a similar manner, and put $1/P = y$;

$$\text{and now } \frac{1}{y} - a = b \cos x + c \sin x,$$

$$\text{so that } \frac{1}{y^2} \frac{dy}{dx} = b \sin x - c \cos x = \sqrt{\left\{b^2 + c^2 - \left(\frac{1}{y} - a\right)^2\right\}};$$

$$\text{and } \int \frac{dx}{a + b \cos x + c \sin x} = \int \frac{dy}{\sqrt{\{(-a^2 + b^2 + c^2)y^2 + 2ay - 1\}}},$$

which by the results of § 191 can be expressed by

$$\begin{aligned} \text{(i.) } & \frac{1}{\sqrt{(a^2 - b^2 - c^2)}} \cos^{-1} \frac{a + (-a^2 + b^2 + c^2)y}{\sqrt{(b^2 + c^2)}} \\ &= \frac{1}{\sqrt{(a^2 - b^2 - c^2)}} \cos^{-1} \frac{aP - a^2 + b^2 + c^2}{\sqrt{(b^2 + c^2)P}}; \end{aligned}$$

or (ii.) $\frac{1}{\sqrt{(-a^2+b^2+c^2)}} \cosh^{-1} \frac{aP - a^2 + b^2 + c^2}{\sqrt{(b^2+c^2)P}};$

as in ex. 34, p. 81.

Similarly to integrate $1/Q$, where

$$Q = a + b \cosh x + c \sinh x,$$

we put $1/Q = y$; and obtain the results of ex. 35, p. 81.

Between the limits 0 and π , (i.) $= \pi/\sqrt{(a^2-b^2-c^2)}$ but (ii.) is illusory, as the function to be integrated becomes infinite between the limits of integration.

Again, between the limits 0 and ∞ ,

$$\int dx/Q \text{ or } \int dx/(a + b \cosh x + c \sinh x)$$

is given by means of ex. 35, p. 81, as

$$\frac{2}{\sqrt{(-a^2+b^2-c^2)}} \tan^{-1} \frac{\sqrt{(-a^2+b^2-c^2)}}{a+b+c},$$

or $\frac{2}{\sqrt{(a^2-b^2+c^2)}} \tanh^{-1} \left\{ \frac{\sqrt{(a^2-b^2+c^2)}}{a+b+c} \right\}.$

200. The integration of $(a+b \cos \theta)^{-n}$ is effected

(i.) when $b/a < 1$, by the substitution

$$a + b \cos \theta = \frac{a^2 - b^2}{a - b \cos v}, \text{ or } \tan \frac{1}{2} \theta = \sqrt{\left(\frac{a+b}{a-b}\right)} \tan \frac{1}{2} v,$$

equivalent geometrically to a change from the true anomaly θ to the excentric anomaly v (§ 177) in an ellipse of excentricity b/a ; and now

$$\int \frac{d\theta}{(a+b \cos \theta)^n} = \frac{1}{(a^2-b^2)^{n-\frac{1}{2}}} \int (a-b \cos v)^{n-1} dv;$$

(ii.) when $b/a > 1$, by the substitution

$$a + b \cos \theta = \frac{a^2 - b^2}{a - b \cosh u}, \text{ or } \tan \frac{1}{2} \theta = \sqrt{\left(\frac{b+a}{b-a}\right)} \tanh \frac{1}{2} u,$$

a change from the true anomaly θ to the hyperbolic excentric anomaly u in a hyperbolic orbit of excentricity b/a ; and now

$$\int \frac{d\theta}{(a+b \cos \theta)^n} = \frac{1}{(b^2-a^2)^{n-\frac{1}{2}}} \int (b \cosh u - a)^{n-1} du.$$

When $n=1$, we obtain, as in ex. 31, p. 80,

$$\begin{aligned} \int \frac{d\theta}{a+b \cos \theta} &= \frac{v}{\sqrt{(a^2-b^2)}} = \frac{1}{\sqrt{(a^2-b^2)}} \cos^{-1} \frac{a \cos \theta + b}{a+b \cos \theta}, \\ \text{or} \quad &= \frac{u}{\sqrt{(b^2-a^2)}} = \frac{1}{\sqrt{(b^2-a^2)}} \cosh^{-1} \frac{a \cos \theta + b}{a+b \cos \theta}. \end{aligned}$$

The integral for $n=2$ is required, as in § 180, for the mean anomaly nt in an elliptic or hyperbolic orbit.

Reciprocally, since $\cos \theta = (a \cosh u - b)/(a - b \cosh u)$,

$$\begin{aligned} \int \frac{du}{(b \cosh u - a)^n} &= \frac{1}{(b^2-a^2)^{n-\frac{1}{2}}} \int (a+b \cos \theta)^{n-1} d\theta, \\ \int \frac{du}{(b-a \cosh u)^n} &= \frac{1}{(b^2-a^2)^{n-\frac{1}{2}}} \int (a+b \cos \theta)^{n-1} (\sec \theta)^n d\theta; \end{aligned}$$

including all possible cases required in the integration of
 $(\alpha + \beta \cosh u)^{-n}$.

The results for $n=1$ will be found in ex. 32, p. 80.

To integrate $(a+b \sinh u)^{-n}$, substitute

$$a+b \sinh u = (a^2+b^2)/(a-b \sinh v);$$

and now

$$\int \frac{du}{(a+b \sinh u)^n} = \frac{1}{(a^2+b^2)^{n-\frac{1}{2}}} \int (a-b \sinh v)^{n-1} dv.$$

If we put $\theta = \operatorname{gd} u$, $\phi = \operatorname{gd} v$, and $b/a = \tan \gamma$,

$$\text{then} \quad \sinh u = \tan \theta = \frac{a \tan \phi + b}{a - b \tan \phi} = \tan(\phi + \gamma);$$

so that $\theta = \phi + \gamma$; and now

$$\int \frac{du}{(a+b \sinh u)^n} = \frac{1}{(a^2+b^2)^{\frac{1}{2}n}} \int (\cos \theta)^{n-1} (\sec \phi)^n d\theta \quad (\text{or } d\phi)$$

and the result for $n=1$ is given in ex. 33, p. 81.

With the notation of the confocal conics of § 177,

$$\tan \frac{1}{2}\theta = \coth \frac{1}{2}u \tan \frac{1}{2}v, \quad \tan \frac{1}{2}\theta' = \tanh \frac{1}{2}u \tan \frac{1}{2}v;$$

and taking the logarithmic differentials

$$\frac{d\theta}{\sin \theta} = \frac{-du}{\sinh u} = \frac{dv}{\sin v}, \quad \frac{d\theta'}{\sin \theta'} = \frac{du}{\sinh u} = \frac{dv}{\sin v};$$

while

$$\begin{aligned} (\cosh u - \cos v)(\cosh u + \cos \theta) \\ &= (\cosh u + \cos v)(\cosh u - \cos \theta') = \sinh^2 u, \\ (\cosh u - \cos v)(\cos v - \cos \theta) \\ &= (\cosh u + \cos v)(\cos \theta' - \cos v) = \sin^2 v; \end{aligned}$$

so that

$$\begin{aligned} \int_0^{\pi} \frac{(\sinh u)^{2n-1} dv}{(\cosh u + \cos v)^n} &= \int_0^{\pi} (\cosh u - \cos \theta')^n d\theta', \\ \int_0^{\pi} \frac{(\sin v)^{2n-1} du}{(\cosh u + \cos v)^n} &= \int_0^{\pi} (\cos \theta' - \cos v)^n d\theta'; \text{ etc.} \end{aligned}$$

As applications the student may evaluate the expression for the area $PpP'p'$ bounded by the elliptic arcs u, u' and the hyperbolic arcs v, v' of the confocals of § 177, or bounded by the circular arcs u, u' and v, v' of the system of dipolar circles of § 175, given by the integral

$$\iint \frac{\partial(x, y)}{\partial(u, v)} du dv = \frac{1}{2}c^2 \iint (\cosh 2u - \cos 2v) du dv,$$

or $\iint \frac{c^2 du dv}{(\cosh v + \cos u)^2}$; and determine the centroid.

Prove also that the c.g. is at N (fig. 54, i.) of

(i.) a circular wire in which the line density varies as SP^{-2} ; (ii.) a circular area in which the surface density varies as SP^{-4} ; (iii.) a spherical shell in which the surface density varies as SP^{-3} ;

(iv.) a solid sphere in which the volume density varies as SP^{-5} ; the boundary in each case being defined by $u = a$ positive constant, so as to include N in the boundary.

Determine also the c.g. of a quadrant of an elliptic wire, as in fig. 55, in which the line density varies as p , $1/p$, xp , x/p , ..., p denoting the length of the perpendicular from the centre O on the tangent.

201. The integrals

$$(i.) \int (H \cos \theta + K) d\theta / (A \cos^2 \theta + 2B \cos \theta + C),$$

$$(ii.) \int (H \cosh u + K) du / (A \cosh^2 u + 2B \cosh u + C),$$

$$(iii.) \int (H \sinh u + K) dv / (A \sinh^2 u + 2B \sinh u + C),$$

are reduced to the form of the integral of § 194, by the substitution $x = \cos \theta$, $\cosh u$ or $\sinh v$; and now R is replaced by $\sin \theta$, $\sinh u$, or $\cosh v$; while by writing

$$\sin^2 \theta = \{(1 - \cos \alpha \cos \theta)^2 - (\cos \theta - \cos \alpha)^2\} / \sin^2 \alpha,$$

$$V = A \cos^2 \theta + 2B \cos \theta + C$$

$$= P(1 - \cos \alpha \cos \theta)^2 + Q(\cos \theta - \cos \alpha)^2,$$

and $y = \sin^2 \alpha \sin^2 \theta / V$;

$$1 - Py = (P + Q) / (\cos \theta - \cos \alpha)^2 / V,$$

$$1 + Qy = (P + Q) / (1 - \cos \alpha \cos \theta)^2 / V;$$

then $1/P$ and $-1/Q$ are the maximum and minimum values of y ; and the integration is expressed by inverse sines, circular and hyperbolic, of $\sqrt{(Py)}$ and $\sqrt{(Qy)}$.

Similarly for $\cosh u$ and $\sinh u$, by writing

$$\sinh^2 u = \{(\cosh \beta \cosh u - 1)^2 - (\cosh u - \cosh \beta)^2\} / \sinh^2 \beta,$$

$$\cosh^2 u = \{(\sinh \gamma \sinh u + 1)^2 + (\sinh u - \sinh \gamma)^2\} / \cosh^2 \gamma.$$

As numerical examples, integrate

$$\{L(3 - \cos \theta) + M(1 - 3 \cos \theta)\} / (5 \cos^2 \theta - 6 \cos \theta + 5)$$

$$\{L(\cosh u - 2) + M(2 \cosh u - 1)\} / (5 \cosh^2 u - 8 \cosh u + 5),$$

$$(H \sinh v + K) / (\sinh^2 v - \sinh v + 1),$$

$$(H \sinh v + K) / (6 \sinh^2 v - 4 \sinh v + 9).$$

Examples.—Integrate with respect to x ,

- (1) $\tan x \sec^2 x$, $\tan^2 x \sec^2 x$, $\tan^3 x \sec^2 x$, $(\tan x)^m \sec^2 x$,
 $\cot x \operatorname{cosec}^2 x$, $\cot^2 x \operatorname{cosec}^2 x$, $(\cot x)^m \operatorname{cosec}^2 x$, $\tan x$,
 $\cot x$, $\tan^2 x$, $\cot^2 x$, $\tan^3 x$, $\cot^3 x$, $\tan^4 x$, $\cot^4 x$,
 $\sec x \tan x$, $\sec^2 x \tan x$, $(\sec x)^m \tan x$, $(\operatorname{cosec} x)^m \cot x$,
 $\sec^2 x$, $\sec^4 x$, $\sec^6 x$, $\operatorname{cosec}^6 x$, $\sec x$, $\sec^3 x$, $\operatorname{cosec}^3 x$,
 $\sec x \operatorname{cosec} x$, $\operatorname{vers} x$, $\operatorname{vers}^2 x$, $1/\operatorname{vers} x$; also the corresponding hyperbolic functions of x .

$$(2) \frac{1}{\cos x + \sin x}, \frac{1}{b \cos x + c \sin x}, \frac{1}{a + b \tan x}, \frac{1}{a + b \cos x},$$

$$\frac{1}{5 + 4 \cos x}, \frac{1}{4 + 5 \cos x}, \frac{1}{a^2 \cos^2 x \pm \beta^2 \sin^2 x}, \frac{1}{a^2 \cosh^2 x \pm \beta^2 \sinh^2 x},$$

$$\frac{B \cos x + C \sin x}{b \cos x + c \sin x}, \frac{A + B \cos x + C \sin x}{a + b \cos x + c \sin x},$$

$$\frac{1}{\sin^2 a - \sin^2 x}, \frac{1}{1 - \sin^2 a \sin^2 x}, \sec x \sec 2x, \tan x \tan 2x.$$

- (3) Prove that

$$(i.) \sin x \sec(x-a) \sec(x-b)$$

$$= \cos a \cos b \operatorname{cosec}(a-b) \{ \sec a \sec(x-a) - \sec b \sec(x-b) \};$$

$$(ii.) \frac{\sin x}{\sin(x-a) \sin(x-b) \sin(x-c)} = \sum \frac{\sin a \cot(x-a)}{\sin(a-b) \sin(a-c)};$$

$$(iii.) \frac{\sin^2 x}{\sin(x-a) \sin(x-b) \sin(x-c)} = \sum \frac{\sin^2 a \operatorname{cosec}(x-a)}{\sin(a-b) \sin(a-c)};$$

$$(iv.) \frac{\sin mx}{\sin nx} = \frac{1}{2n} \sum (-1)^k \sin ma \cot \frac{1}{2}(x-a) \quad \left(\begin{array}{l} \text{where } a = k\pi/n, \\ \text{and } m < n \end{array} \right)$$

$$= \frac{1}{2n} \sum (-1)^k \sin ma \operatorname{cosec}(x-a), \text{ or } \frac{1}{2n} \sum (-1)^k \sin ma \cot(x-a),$$

according as $m+n$ is odd or even.

Write down the integrals of these functions.

- (4) Integrate $\sec(x+a) \sqrt{\sec(x+b) \sec(x+c)}$,

by means of ex. 16, p. 78, putting $\tan x = y$.

(5) Prove that the definite integrals

$$(i.) \int_0^{\frac{1}{4}\pi} \tan x dx = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot x dx = \log \sqrt{2} = 0.34657.$$

$$(ii.) \int_0^{\frac{1}{4}\pi} \sec x dx = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \operatorname{cosec} x dx = \log(\sqrt{2} + 1) = 0.88137.$$

$$(iii.) \int_0^{\pi} \frac{dx}{a + b \cos x} = \frac{\pi}{\sqrt{(a^2 - b^2)}}, \text{ if } a > b.$$

$$(iv.) \int_0^{\frac{1}{2}\pi} \frac{dx}{a + b \cos x} = \frac{1}{\sqrt{(a^2 - b^2)}} \cos^{-1} \frac{b}{a},$$

$$\text{or} \quad \frac{1}{\sqrt{(b^2 - a^2)}} \cosh^{-1} \frac{b}{a}.$$

$$(v.) \int_0^{\frac{1}{2}\pi} \frac{d\theta}{5 + 3 \cos \theta} = 0.23180.$$

$$(vi.) \int_0^{\frac{1}{2}\pi} \frac{d\theta}{3 + 5 \cos \theta} = 0.27465.$$

$$(vii.) \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(5 + 3 \cos \theta)^2} = 0.03494.$$

$$(viii.) \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(3 + 5 \cos \theta)^2} = 0.05267.$$

$$(6) (i.) \int_0^{\pi} \frac{H \cos \theta + K}{\cos^2 \theta - 2 \sin a \cos \theta + 1} d\theta = \frac{H \sin \frac{1}{2}a + K \cos \frac{1}{2}a}{\sqrt{(2 \cos^2 a)}} \pi,$$

if $\cos a$ is taken positive.

$$(ii.) \int_0^{\pi} \frac{H \cos \theta + K}{a^2 \cos^2 \theta - 2ab \cos a \cos \theta + b^2} d\theta$$

$$= \frac{H \sqrt{(a^2 + b^2 - c^2)} + K \sqrt{(a^2 + b^2 + c^2)}}{c^2 \sqrt{(-a^2 + b^2 + c^2)}} \pi,$$

where

$$c^4 = a^4 - 2a^2b^2 \cos 2a + b^4.$$

202. *Integration by Parts.*

Corresponding to the formula for the differentiation of a product, by parts (§ 12),

$$\frac{d(uv)}{dx} = \frac{du}{dx}v + u\frac{dv}{dx},$$

is the formula, obtained by integrating both sides,

$$uv = \int \frac{du}{dx}v dx + \int u \frac{dv}{dx} dx,$$

or
$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx}v dx,$$

or, as it may be written,

$$\int u dv = uv - \int v du,$$

the formula of the Integral Calculus, for *integration by parts*, as it is called.

Interpreted geometrically the formula merely asserts that, as in fig. 19, with u and v for x and y , the rectangle $OMPN$ is the sum of the areas OMP , ONP into which the rectangle is divided by any curved line; or that the area ONP , represented by $\int u dv$, is the difference between the rectangle uv , and the area OMP , represented by $\int v du$.

The formula shows how the integral $\int u dv$ may be made to depend upon another integral $\int v du$, which is either integrable, or which is made to depend upon another integral, by another integration by parts.

Thus (i.) to integrate xe^x , take $x = u$, $\frac{dv}{dx} = e^x$; then $v = e^x$, and

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x.$$

(ii.) to integrate $\log x$, take $u = \log x$, $dv = dx$, $v = x$; then $\int \log x dx = x \log x - \int x dx/x = x \log x - x$.

(iii.) $\sin \sqrt{x}$, put $\sqrt{x} = z$, then

$$\int \sin \sqrt{x} dx = \int \sin z \cdot 2z dz$$

$$\begin{aligned}
 &= -2z \cos z + 2 \int \cos z dz = -2z \cos z + 2 \sin z \\
 &= -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x}.
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv.) } \int \sin^{-1} x dx &= x \sin^{-1} x - \int \frac{x dx}{\sqrt{(1-x^2)}} \\
 &= x \sin^{-1} x + \sqrt{(1-x^2)}.
 \end{aligned}$$

(v.) to integrate $\cos mx \cosh nx$; here $\cos mx$ or $\cosh nx$ may be taken indiscriminately for u , and we must integrate by parts twice.

$$\text{For instance if } \cos mx = u, \cosh nx = \frac{dv}{dx}, v = \frac{\sinh nx}{n},$$

$$\begin{aligned}
 \int \cos mx \cosh nx dx &= \frac{1}{n} \cos mx \sinh nx + \frac{m}{n} \int \sin mx \sinh nx dx \\
 &= \frac{1}{n} \cos mx \sinh nx + \frac{m}{n^2} \sin mx \cosh nx - \frac{m^2}{n^2} \int \cos mx \cosh nx dx;
 \end{aligned}$$

and therefore, by transposition and division,

$$\int \cos mx \cosh nx dx = \frac{m \sin mx \cosh nx + n \cos mx \sinh nx}{m^2 + n^2} \quad \text{(i.)}$$

In a similar manner

$$\int \cos mx \sinh nx dx = \frac{m \sin mx \sinh nx + n \cos mx \cosh nx}{m^2 + n^2} \quad \text{(ii.)}$$

$$\int \sin mx \cosh nx dx = \frac{n \sin mx \sinh nx - m \cos mx \cosh nx}{m^2 + n^2} \quad \text{(iii.)}$$

$$\int \sin mx \sinh nx dx = \frac{n \sin mx \cosh nx - m \cos mx \sinh nx}{m^2 + n^2} \quad \text{(iv.)}$$

By addition of (i.) and (ii.), (iii.) and (iv.), or independently by integration by parts,

$$\int e^{nx} \cos mx dx = e^{nx} \frac{n \cos mx + m \sin mx}{m^2 + n^2} \dots \dots \dots \text{(v.)}$$

$$\int e^{nx} \sin mx dx = e^{nx} \frac{n \sin mx - m \cos mx}{m^2 + n^2} \dots \dots \dots \text{(vi.)}$$

With a subsidiary angle α given by $\tan \alpha = m/n$,

$$\int e^{nx} \cos mx dx = e^{nx} \cos(mx - \alpha) \cos \alpha / n,$$

$$\int e^{nx} \sin mx dx = e^{nx} \sin(mx - \alpha) \cos \alpha / n.$$

Again, by successive integration by parts,

$$\int e^{x/a} u dx = a e^{x/a} \left(u - a \frac{du}{dx} + a^2 \frac{d^2 u}{dx^2} - a^3 \frac{d^3 u}{dx^3} + \dots \right).$$

$$\begin{aligned} \int u \frac{d^n v}{dx^n} dx &= u \frac{d^{n-1} v}{dx^{n-1}} - \frac{du}{dx} \frac{d^{n-2} v}{dx^{n-2}} + \dots \\ &\quad + (-1)^{n-1} \frac{d^{n-1} u}{dx^{n-1}} v + (-1)^n \int \frac{d^n u}{dx^n} v dx; \end{aligned}$$

the integrated terms of which can again be integrated by parts.

We can establish Taylor's Theorem by successive integration by parts; for

$$\begin{aligned} f(a+h) - fa &= \int_a^{a+h} f'x dx; \\ &= - \left\{ (a+h-x)f'x \right\}_a^{a+h} + \int_a^{a+h} (a+h-x)f''x dx \\ &= hf'a - \left\{ \frac{1}{2!} (a+h-x)^2 f''x \right\}_a^{a+h} + \frac{1}{2!} \int_a^{a+h} (a+h-x)^2 f'''x dx \\ &= hf'a + \frac{h^2}{2!} f''a + \dots + \frac{h^n}{n!} f^n a + \frac{1}{n!} \int_a^{a+h} (a+h-x)^n f^{n+1}x dx; \end{aligned}$$

so that, as in § 114,

$$\begin{aligned} f(a+h) - fa - hf'a - \frac{h^2}{2!} f''a - \dots - \frac{h^n}{n!} f^n a \\ = R = \frac{1}{n!} \int_a^{a+h} (a+h-x)^n f^{n+1}x dx. \end{aligned}$$

It is assumed here that $fx, f'x, f''x, \dots f^n x, f^{n+1}x$ are all finite between the limits a and $a+h$ of x ; and now if P denotes a certain average value of $f^{n+1}x$ between the limits a and $a+h$, which we may denote by $f^{n+1}(a+\theta h)$,

$$R = \frac{1}{n!} \int_a^{a+h} (a+h-x)^n P dx = \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta h).$$

Examples.—Integrate by parts,

$$x^2 e^x, x^3 e^{ax}, x^4 e^{ax+b}, \dots;$$

$$e^{\sqrt{x}}, e^{\sqrt[3]{x}}, \exp \sqrt[n]{x};$$

$$x^m \log x^n, (\log x)^2, (\log x)^3, (\log x)^4, (\log x)^5, \dots;$$

$$x^2 \cos x, x^3 \sin x, \dots; \cos \sqrt[3]{x}, \sin \sqrt[4]{x}, \dots;$$

$$e^{ax+b} \cos(mx+n), e^{ax+b} \sin(mx+n), \dots;$$

$$\cos^{-1}x, \tan^{-1}x, \cot^{-1}x, \sec^{-1}x, \operatorname{cosec}^{-1}x, \operatorname{vers}^{-1}x;$$

$$\cosh^{-1}x, \sinh^{-1}x, \tanh^{-1}x, \dots;$$

$$(\sin^{-1}x)^2, (\sinh^{-1}x)^2, (\cosh^{-1}x)^2, (\operatorname{vers}^{-1}x)^2;$$

$$\tan^{-1} \sqrt{x}, \tanh^{-1} \sqrt[3]{x}, \dots;$$

$$\sqrt{(a^2 - x^2)}, \sqrt{(x^2 - a^2)}, \sqrt{(a^2 + x^2)}, (a^2 - x^2)^{\frac{3}{2}}, \\ x^2 \sqrt{(a^2 - x^2)}, \dots;$$

$$x \sin^{-1}x, x^2 \cos^{-1}x, x^3 \cosh^{-1}x \dots;$$

$$\log x \cdot \sin^{-1}x, \log x \cdot \cosh^{-1}x;$$

$$(\sec x)^3, (\sec x)^5, (\operatorname{cosec} x)^5, \dots \sec x \tan^2 x, \sec x \tan^4 x, \dots$$

203. The integration of $\sqrt{(a^2 - x^2)}$, $\sqrt{(x^2 - a^2)}$, $\sqrt{(a^2 + x^2)}$ is required in the quadrature of the circle, ellipse, and hyperbola (§§ 50-54); thus for example, integrating by parts for the quadrature of the circle,

$$\int \sqrt{(a^2 - x^2)} dx = x \sqrt{(a^2 - x^2)} + \int \frac{x^2 dx}{\sqrt{(a^2 - x^2)}} \\ = x \sqrt{(a^2 - x^2)} + a^2 \int \frac{dx}{\sqrt{(a^2 - x^2)}} - \int \sqrt{(a^2 - x^2)} dx,$$

(and, by transposition and division)

$$= \frac{1}{2} \sqrt{(a^2 - x^2)} + \frac{1}{2} a^2 \sin^{-1}(x/a) \quad (\S 50).$$

Similarly
$$\int \sqrt{(a-x)(x-\beta)} dx$$

$$= \frac{1}{2} \left(x - \frac{a+\beta}{2} \right) \sqrt{(a-x)(x-\beta)} + \frac{1}{4} (a-\beta)^2 \tan^{-1} \frac{x-\beta}{a-x};$$

so that taken between the limits a and β , at which $\sqrt{(a-x)(x-\beta)}$ vanishes, and between which it is real, the integral is $\frac{1}{8}\pi(a-\beta)^2$.

More generally, denoting $ax^2+2bx+c$ by R , and integrating by parts

$$\int \sqrt{R} dx = \frac{1}{2} \frac{ax+b}{a} \sqrt{R} - \frac{1}{2} \frac{b^2-ac}{a} \int \frac{dx}{\sqrt{R}};$$

and the result has one of the three forms of ex. 28, p. 80.

When a is negative and b^2-ac positive, the value of x is restricted to lie between the roots, α and β suppose, of the equation $R=0$, for R to be positive and \sqrt{R} real; and now, taken between the limits α and β ,

$$\int_{\alpha}^{\beta} \sqrt{R} dx = \frac{ac-b^2}{\sqrt{(-a^3)}} \frac{\pi}{2}.$$

Suppose for instance that the area of the ellipse is required, when it is given by the general equation of the second degree (§ 13)

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Solving this equation as a quadratic in y ,

$$hx + by + f = \sqrt{\{(h^2-ab)x^2 + 2(fh-bg)x + f^2-bc\}};$$

so that (§ 49)

$$b(y_1 - y_2) = 2\sqrt{\{(h^2-ab)x^2 + 2(fh-bg)x + f^2-bc\}};$$

and the area of the ellipse is $\int (y_1 - y_2) dx$, taken between the limits for which the quantity under the radical is positive; and therefore the area is $\pi\Delta/(ab-h^2)^{\frac{3}{2}}$, where Δ denotes the *discriminant* (ex. 9, p. 139).

This result must be multiplied by $\sin \omega$, if the co-ordinate axes are inclined at an angle ω .

204. *Formulas of Reduction.*

A *formula of reduction* in the Integral Calculus is a formula, obtained in general by integration by parts, by which one integral, say u_n , is made to depend upon a simpler integral, say u_{n-1} , or u_{n-2} ; then by successive substitution in the formula of reduction, we finally arrive at an integration which can be effected.

Suppose, for instance, $u_n = \int (\sin \theta)^n d\theta$; integrating by parts, with $u = (\sin \theta)^{n-1}$, $dv = \sin \theta d\theta$, $v = -\cos \theta$,

$$\begin{aligned} u_n &= \int (\sin \theta)^n d\theta = \int (\sin \theta)^{n-1} \sin \theta d\theta \\ &= -(\sin \theta)^{n-1} \cos \theta + (n-1) \int (\sin \theta)^{n-2} (\cos \theta)^2 d\theta \\ &= \dots\dots\dots + (n-1) u_{n-2} - (n-1) u_n; \end{aligned}$$

$$\text{or} \quad n u_n = \dots\dots\dots + (n-1) u_{n-2},$$

$$u_n = -\frac{1}{n} (\sin \theta)^{n-1} \cos \theta + \frac{n-1}{n} u_{n-2},$$

a *formula of reduction*.

Similarly

$$\int (\cos \theta)^n d\theta = \frac{1}{n} \sin \theta (\cos \theta)^{n-1} + \frac{n-1}{n} \int (\cos \theta)^{n-2} d\theta,$$

another formula of reduction.

Taken between the limits 0 and $\frac{1}{2}\pi$,

$$\int_0^{\frac{1}{2}\pi} (\sin \theta)^n d\theta = \frac{n-1}{n} \int_0^{\frac{1}{2}\pi} (\sin \theta)^{n-2} d\theta,$$

$$\int_0^{\frac{1}{2}\pi} (\cos \theta)^n d\theta = \frac{n-1}{n} \int_0^{\frac{1}{2}\pi} (\cos \theta)^{n-2} d\theta.$$

First suppose n is an even integer $2m$; then

$$\begin{aligned} \int_0^{\frac{1}{2}\pi} \left(\frac{\sin \theta}{\cos \theta} \right)^{2m} d\theta &= \frac{2m-1}{2m} \int_0^{\frac{1}{2}\pi} \left(\frac{\sin \theta}{\cos \theta} \right)^{2m-2} d\theta \\ &= \frac{2m-3}{2m-2} \frac{2m-1}{2m} \int_0^{\frac{1}{2}\pi} \left(\frac{\sin \theta}{\cos \theta} \right)^{2m-4} d\theta, \end{aligned}$$

and so on, finally being

$$\int_0^{\frac{1}{2}\pi} \left(\frac{\sin \theta}{\cos \theta} \right)^{2m} d\theta = \frac{1 \cdot 3 \cdot 5 \dots 2m-1}{2 \cdot 4 \cdot 6 \dots 2m} \frac{\pi}{2}.$$

Secondly, suppose n an odd integer $2m+1$; then

$$\begin{aligned} \int_0^{\frac{1}{2}\pi} \left(\frac{\sin \theta}{\cos \theta} \right)^{2m+1} d\theta &= \frac{2 \cdot 4 \cdot 6 \dots 2m}{3 \cdot 5 \cdot 7 \dots 2m+1} \int_0^{\frac{1}{2}\pi} \frac{\sin \theta}{\cos \theta} d\theta. \\ &= \frac{2 \cdot 4 \cdot 6 \dots 2m}{3 \cdot 5 \cdot 7 \dots 2m+1}. \end{aligned}$$

These are called *Wallis's Theorems*, and are of great practical use in the Integral Calculus.

Both cases are included in the formula

$$\int_0^{\frac{1}{2}\pi} \left(\frac{\sin \theta}{\cos \theta} \right)^n d\theta = \frac{(n-1)(n-3) \dots (\frac{1}{2}\pi)^{\frac{1}{2} + \frac{1}{2}(-1)^n}}{n(n-2)(n-4) \dots}$$

If we put $\cos \theta = \operatorname{sech} u$, or $\theta = \operatorname{gd} u$, then $d\theta = \operatorname{sech} u du$;

and $\int_0^{\frac{1}{2}\pi} (\cos \theta)^n d\theta = \int_0^\infty (\operatorname{sech} u)^{n+1} du$; so that

$$\begin{aligned} \int_0^\infty (\operatorname{sech} u)^{2m+1} du &= \frac{1 \cdot 3 \cdot 5 \dots 2m-1}{2 \cdot 4 \cdot 6 \dots 2m} \frac{\pi}{2}; \\ \int_0^\infty (\operatorname{sech} u)^{2m} du &= \frac{2 \cdot 4 \cdot 6 \dots 2m-2}{3 \cdot 5 \cdot 7 \dots 2m-1}. \end{aligned}$$

When m or n is not an integer, these integrals depend upon the Gamma Function, defined in the next article.

Examples.—Prove the following formulas of reduction, and evaluate the integrals for $n=1, 2, 3, 4, \dots$

$$(1) \quad u_n = \int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx = x^n e^x - n u_{n-1}.$$

$$(2) \quad u_n = \int (\log x)^n dx = x(\log x)^n - n u_{n-1}.$$

$$(3) \quad u_n = \int x^m (\log x)^n dx = \frac{x^{m+1} (\log x)^n}{m+1} - \frac{n u_{n-1}}{m+1}.$$

$$(4) \quad u_n = \int x^n \sqrt{(a^2 - x^2)} dx = -\frac{x^{n-1}(a^2 - x^2)^{\frac{3}{2}}}{n+2} + \frac{n-1}{n+2} a^2 u_{n-2}.$$

$$(5) \quad u_n = \int x^n \sqrt{(2ax - x^2)} dx \\ = -\frac{x^{n-1}(2ax - x^2)^{\frac{3}{2}}}{n+2} + \frac{2n+1}{n+2} a u_{n-1}.$$

$$(6) \quad u_n = \int (a^2 - x^2)^{n-\frac{1}{2}} = \{x(a^2 - x^2)^{n-\frac{1}{2}} + (2n-1)a^2 u_{n-1}\} / 2n.$$

$$(7) \quad u_n = \int \frac{dx}{(Ax^2 + 2Bx + C)^n} \\ = \frac{1}{2n-2} \frac{1}{AC - B^2} \frac{Ax + B}{(Ax^2 + 2Bx + C)^{n-1}} + \frac{2n-3}{2n-2} \frac{A u_{n-1}}{AC - B^2}.$$

$$(8) \quad u_n = \int \frac{dx}{(x^p + a^p)^n} = \frac{x}{(n-1)p a^p (x^p + a^p)^{n-1}} + \frac{np - p - 1}{(n-1)p} \frac{u_{n-1}}{a^p}.$$

$$(9) \quad u_n = \int \frac{dx}{(x-a)^n \sqrt{(x-b)}} \\ = \frac{1}{n-1} \frac{1}{b-a} \frac{\sqrt{(x-b)}}{(x-a)^{n-1}} + \frac{2n-3}{2n-2} \frac{u_{n-1}}{b-c}.$$

$$(10) \quad u_n = \int x^n dx / \sqrt{R} \\ = \{x^{n-1} \sqrt{R} - (2n-1)bu_{n-1} - (n-1)cu_{n-2}\} / na.$$

(11) Determine L, M, N in the formulas of reduction

$$u_n = \int \frac{dx}{x^n \sqrt{R}} = \frac{L \sqrt{R}}{x^{n-1}} + M u_{n-1} + N u_{n-2}.$$

$$(12) \quad u_n = \int \frac{dx}{(x-q)^n \sqrt{R}} = \frac{L \sqrt{R}}{(x-q)^{n-1}} + M u_{n-1} + N u_{n-2}.$$

$$(13) \quad u_n = \int (\tan x)^n dx = \frac{(\tan x)^{n-1}}{n-1} - u_{n-2}.$$

$$(14) \quad u_n = \int (\cot x)^n dx = -\frac{(\cot x)^{n-1}}{n-1} - u_{n-2}.$$

$$(15) \quad u_n = \int (\sec x)^n dx = \frac{(\sec x)^{n-2} \tan x}{n-1} + \frac{n-2}{n-1} u_{n-2}.$$

$$(16) \quad u_n = \int (\operatorname{cosec} x)^n dx = -\frac{(\operatorname{cosec} x)^{n-2} \cot x}{n-1} + \frac{n-2}{n-1} u_{n-2}.$$

$$(17) \quad u_n = \int (\text{vers } x)^n dx = -\frac{\sin x (\text{vers } x)^{n-1}}{n} + \frac{2n-1}{n} u_{n-1}.$$

$$(18) \quad u_n = \int x^n \cos mx = \frac{x^n}{m} \sin mx + \frac{nx^{n-1}}{m^2} \cos mx - \frac{n(n-1)}{m^2} u_{n-2}.$$

$$(19) \quad u_n = \int e^{px} (\cos x)^n dx = \frac{e^{px} (\cos x)^{n-1} (p \cos x + n \sin x)}{n^2 + p^2} + \frac{n(n-1)}{n^2 + p^2} u_{n-2}.$$

$$(20) \quad u_{(m, n)} = \int (\sin x)^m (\cos x)^n dx \\ = \frac{(\sin x)^{m+1} (\cos x)^{n-1}}{m+n} + \frac{n-1}{m+n} u_{(m, n-2)},$$

$$\text{or} \quad -\frac{(\sin x)^{m-1} (\cos x)^{n+1}}{m+n} + \frac{m-1}{m+n} u_{(m-2, n)},$$

$$\text{or} \quad \frac{(m-1)(\sin x)^{m+1} (\cos x)^{n-1} - (n-1)(\sin x)^{m-1} (\cos x)^{n+1}}{(m+n)(m+n-2)} + \frac{(m-1)(n-1)u_{(m-2, n-2)}}{(m+n)(m+n-2)},$$

$$(21) \quad \int_0^{\frac{1}{2}\pi} (\sin x)^m (\cos x)^{2n+1} dx = \frac{2 \cdot 4 \cdot 6 \dots 2n}{(m+1)(m+3) \dots (m+2n+1)}.$$

$$(22) \quad \int_0^{\frac{1}{2}\pi} (\sin x)^{2m} (\cos x)^{2n} dx = \frac{1 \cdot 3 \cdot 5 \dots (2m-1) \cdot 1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2m(2m+2) \dots (2m+2n)} \frac{\pi}{2}.$$

$$(23) \quad u_n = \int \frac{d\theta}{(a + b \cos \theta)^n} = -\frac{b \sin \theta}{(n-1)(a^2 - b^2)(a + b \cos \theta)^{n-1}} + \frac{(2n-3)au_{n-1} - (n-2)u_{n-2}}{(n-1)(a^2 - b^2)}.$$

Investigate formulas of reduction for the integration of $(\sin^{-1}x)^n$, $(\text{vers}^{-1}x)^n$, $(\cosh x)^n$, $(\sinh x)^n$, $(\tanh x)^n$, $(\text{sech } x)^n$, $(\sinh^{-1}x)^n$, $(\cosh^{-1}x)^n$, $\sin \sqrt[n]{x}$, $\sinh \sqrt[n]{x}$, $(a + b \cosh u)^{-n}$, $(a + b \cos \theta + c \sin \theta)^{-n}$, $(a + b \cosh u + c \sinh u)^{-n}$.

205. *Definite Integrals.*

Any integral taken between definite fixed limits, is called a *definite* integral (§ 45), but the name is used especially for a large class of integrals which can only be evaluated between certain limits, and of which the *indefinite* integrals (§ 39) cannot be found.

The subject of Definite Integrals is a large one, and would lead us beyond the scope of the present treatise; we may however consider a few elementary definite integrals, which are of frequent occurrence, and great practical use.

Consider first the definite integral

$$\int_0^{\infty} e^{-x} x^{n-1} dx, \text{ or } \int_0^1 (-\log z)^{n-1} dz = \int_0^1 (\log 1/z)^{n-1} dz,$$

if $e^{-x} = z$; this is called the Gamma function; because it is usually denoted by Γn .

By integration by parts

$$\Gamma n = \left(-e^{-x} x^{n-1} \right)_0^{\infty} + (n-1) \int_0^{\infty} e^{-x} x^{n-2} dx = (n-1) \Gamma(n-1);$$

so that if n is a positive integer,

$$\Gamma n = (n-1)! \Gamma 1 = (n-1)!$$

since $\Gamma 1 = \int_0^{\infty} e^{-x} dx = 1$, also $\Gamma 0 = \infty$.

Changing x into x^2 gives

$$\Gamma m = 2 \int_0^{\infty} e^{-x^2} x^{2m-1} dx;$$

so that we may express the product of Γm and Γn by

$$\Gamma m \Gamma n = 4 \int_0^{\infty} e^{-x^2} x^{2m-1} dx \int_0^{\infty} e^{-y^2} y^{2n-1} dy$$

$$\begin{aligned}
&= 4 \int_0^\infty \int_0^\infty e^{-x^2-y^2} x^{2m-1} y^{2n-1} dx dy \\
&= 4 \int_0^\infty e^{-r^2} r^{2m+2n-1} dr \int_0^{\frac{1}{2}\pi} (\sin \theta)^{2m-1} (\cos \theta)^{2n-1} d\theta
\end{aligned}$$

(on changing to polars, with $x = r \cos \theta$, $y = r \sin \theta$)

$$= 2\Gamma(m+n) \int_0^{\frac{1}{2}\pi} (\sin \theta)^{2m-1} (\cos \theta)^{2n-1} d\theta.$$

The definite integral

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx = 2 \int_0^{\frac{1}{2}\pi} (\sin \theta)^{2m-1} (\cos \theta)^{2n-1} d\theta,$$

on putting $x = (\sin \theta)^2$; and is called the *First Eulerian Integral*, and denoted by $B(m, n)$; so that

$$\Gamma m \Gamma n = \Gamma(m+n) B(m, n).$$

Also, on substituting

$$\begin{aligned}
&a - x = (a - \beta) \sin^2 \theta, \quad x - \beta = (a - \beta) \cos^2 \theta, \\
&\int_\beta^a (a-x)^{m-1} (x-\beta)^{n-1} dx = (a-\beta)^{m+n-1} B(m, n).
\end{aligned}$$

This integral has been evaluated for positive integral values of m and n (p. 424); and for fractional values it depends on the tabulated values of the Gamma Function, by means of the above relation

$$B(m, n) = \Gamma m \Gamma n / \Gamma(m+n).$$

When $m+n=1$, then (§ 189)

$$\Gamma m \Gamma(1-m) = B(m, 1-m) = 2 \int_0^{\frac{1}{2}\pi} (\tan \theta)^{2m-1} d\theta = \pi \operatorname{cosec} m\pi.$$

Putting $m = \frac{1}{2}$, gives $\Gamma \frac{1}{2} = \sqrt{\pi}$, and $\Gamma \frac{3}{2} = \frac{1}{2} \Gamma \frac{1}{2} = \frac{1}{2} \sqrt{\pi}$; and now if Γm is calculated from $m=0.5$ to $m=1$, then Γm is known from $m=0.5$ to $m=0$; and thence Γn or $\log \Gamma n$ can be tabulated from $n=1$ to $n=2$ (Bertrand, *Calcul Intégral*, p. 285), and the graph of Γx can be drawn for all positive values of x .

The substitution of $x=z/(1+z)$ or $1/(1+z)$ shows that

$$B(m, n) = \int_0^{\infty} \frac{z^{m-1} dz}{(1+z)^{m+n}} = \int_0^1 \frac{z^{n-1} dz}{(1+z)^{m+n}} = B(n, m).$$

As an application, let us determine the area A of the positive quadrant of the curve $x^n + y^n = a^n$; then

$$A = \int_0^a (a^n - x^n)^{\frac{1}{n}} dx = \frac{2}{n} a^2 \int_0^{\frac{1}{2}\pi} (\sin \theta)^{\frac{2}{n}-1} (\cos \theta)^{\frac{2}{n}+1} d\theta$$

(on substituting $x = a^n \sin^2 \theta$)

$$\begin{aligned} &= \frac{a^2}{n} B\left(\frac{1}{n}, \frac{1}{n} + 1\right) = \frac{a^2}{n} \Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{1}{n} + 1\right) / \Gamma\left(\frac{2}{n} + 1\right) \\ &= \frac{a^2}{2n} \left(\Gamma\left(\frac{1}{n}\right)\right)^2 / \Gamma\left(\frac{2}{n}\right). \end{aligned}$$

For instance, if $n=2$, we obtain the area of the quadrant of the circle $\frac{1}{4}\pi a^2$.

$$\text{Also} \quad \bar{x} = \bar{y} = \frac{2}{3} a \left(\Gamma\left(\frac{2}{n}\right)\right)^2 / \left(\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{3}{n}\right)\right).$$

By integrating, as in § 134, throughout the space in which x, y, z are positive, but

$$(x/a)^m + (y/b)^n + (z/c)^p - 1$$

is negative, we can show in a similar manner, that the volume V of this space within the surface is given by

$$\frac{V}{abc} = \Gamma\left(\frac{1}{m} + 1\right) \Gamma\left(\frac{1}{n} + 1\right) \Gamma\left(\frac{1}{p} + 1\right) / \Gamma\left(\frac{1}{m} + \frac{1}{n} + \frac{1}{p} + 1\right)$$

while

$$\frac{\bar{x}V}{a^2bc} = \frac{1}{2} \Gamma\left(\frac{2}{m} + 1\right) \Gamma\left(\frac{1}{n} + 1\right) \Gamma\left(\frac{1}{p} + 1\right) / \Gamma\left(\frac{2}{m} + \frac{1}{n} + \frac{1}{p} + 1\right), \dots$$

agreeing with § 134 when we put $m=n=p=2$.

Again, the length of a branch of the curve $r^n = a^n \cos n\theta$

(§ 172) can be expressed by $\frac{a}{n} B\left(\frac{1}{2n}, \frac{1}{2}\right)$.

206. Bernoulli's and Euler's Numbers as Definite Integrals.

Making use of the relation, with $mu = x$,

$$\int_0^{\infty} e^{-mu} u^{p-1} du = m^{-p} \int_0^{\infty} e^{-x} x^{p-1} dx = m^{-p} \Gamma p,$$

then, with the notation of § 112,

$$\begin{aligned} T_p \Gamma p &= \int_0^{\infty} (e^{-u} + e^{-3u} + e^{-5u} + \dots) u^{p-1} du \\ &= \int_0^{\infty} \frac{e^{-u} u^{p-1} du}{1 - e^{-2u}} = \frac{1}{2} \int_0^{\infty} \operatorname{cosech} u \cdot u^{p-1} du; \end{aligned}$$

so that
$$S_p = \frac{2^p T_p}{2^p - 1} = \frac{2^{p-1}}{(2^p - 1) \Gamma p} \int_0^{\infty} \operatorname{cosech} u \cdot u^{p-1} du;$$

and
$$B_n = \frac{2n}{(2^{2n} - 1) \pi^{2n}} \int_0^{\infty} \operatorname{cosech} u \cdot u^{2n-1} du;$$

so that, on putting $u = \frac{1}{2} \pi v$,

$$\int_0^{\infty} \operatorname{cosech} \frac{1}{2} \pi v \cdot v^{2n-1} dv = 2^{2n} (2^{2n} - 1) B_n / 2n,$$

the coefficient of $x^{2n-1} / (2n-1)!$ in the expansion of $\tan x$, and this is always an integer; namely 1, 2, 16, 272,

Similarly (§§ 112, 113)

$$\begin{aligned} U_p \Gamma p &= \int_0^{\infty} (e^{-u} - e^{-3u} + e^{-5u} \dots) u^{p-1} du = \frac{1}{2} \int_0^{\infty} \operatorname{sech} u \cdot u^{p-1} du; \\ E_n &= 2(2n)! U_{2n+1} / (\frac{1}{2} \pi)^{2n+1} = \int_0^{\infty} \operatorname{sech} \frac{1}{2} \pi v \cdot v^{2n} dv; \end{aligned}$$

which, since E_n is an integer, shows that this definite integral is always an integer, namely 1, 5, 61, 1385,

Put $\operatorname{sech} \frac{1}{2} \pi v = \cos \phi$, $\frac{1}{2} \pi v = \log(\sec \phi + \tan \phi)$,

and we find (§ 113)

$$\int_0^{\frac{1}{2}\pi} \{\log (\sec \phi + \tan \phi)\}^{2n} d\phi = (\tfrac{1}{2}\pi)^{2n+1} E_n = 2(2n)! U_{2n+1},$$

$$\int_0^{\frac{1}{2}\pi} \{\log (\sec \phi + \tan \phi)\}^{2n-1} d\phi = 2(2n-1)! U_{2n}.$$

Thus, for example, the area between a branch of the curve $m\theta = \text{gd } (r/a)$, or $r/a = \log (\sec m\theta + \tan m\theta)$, and its asymptote is

$$\begin{aligned} & \tfrac{1}{2}a^2 \int_0^{\frac{1}{2}\pi/m} \{\log (\sec m\theta + \tan m\theta)\}^2 d\theta \\ &= \tfrac{1}{2}a^2 (\tfrac{1}{2}\pi)^3 E_1/m = \tfrac{1}{18}\pi^3 a^2/m. \end{aligned}$$

207. *Differentiation and Integration of a Definite Integral, and Applications.*

Denoting the indefinite integral with respect to x of a function $f(x, c)$, involving a parameter c (§ 105), by $f_1(x, c)$, then the definite integral

$$I = \int_b^a f(x, c) dx = f_1(a, c) - f_1(b, c);$$

and now, if a, b, c are functions of some variable t ,

$$\begin{aligned} \frac{dI}{dt} &= \frac{\partial I}{\partial a} \frac{da}{dt} + \frac{\partial I}{\partial b} \frac{db}{dt} + \frac{\partial I}{\partial c} \frac{dc}{dt} \\ &= f(a, c) \frac{da}{dt} - f(b, c) \frac{db}{dt} + \int_b^a \frac{\partial f(x, c)}{\partial c} dx \frac{dc}{dt}; \end{aligned}$$

of which a geometrical interpretation can easily be constructed.

We can also integrate I according to a similar rule.

By means of this method of differentiation and integration we can show the connexion between different classes of integrals, as already done in §§ 193, 194, and also verify the solution of certain differential equations, when the solution is given as a definite integral, as in § 184.

Thus, if α and β are positive, and $I = \frac{1}{\sqrt{\alpha\beta}} \tan^{-1} \sqrt{\frac{\beta}{\alpha}}$,

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\alpha \cos^2\theta + \beta \sin^2\theta} = \frac{1}{2\sqrt{\alpha\beta}} \cos^{-1} \left(\frac{\alpha \cos^2\theta - \beta \sin^2\theta}{\alpha \cos^2\theta + \beta \sin^2\theta} \right)_0^{\frac{1}{2}\pi} = I,$$

$$\int_0^\infty \frac{du}{\alpha \cosh^2 u + \beta \sinh^2 u} = \frac{1}{2\sqrt{\alpha\beta}} \cos^{-1} \left(\frac{\alpha \cosh^2 u - \beta \sinh^2 u}{\alpha \cosh^2 u + \beta \sinh^2 u} \right)_0^\infty = I;$$

and
$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{(\alpha \cos^2\theta + \beta \sin^2\theta)^{n+1}} = \frac{(-1)^n}{n!} \left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} \right)^n I,$$

$$\int_0^\infty \frac{du}{(\alpha \cosh^2 u + \beta \sinh^2 u)^{n+1}} = \frac{(-1)^n}{n!} \left(\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta} \right)^n I.$$

We can verify that the definite integral

$$\int_0^\pi \cos(pv - x \sin v) dv,$$

which we have denoted by $\pi J_p(x)$ in § 184, and called the *Bessel function* of x of the p th order, is a solution of Bessel's differential equation given on p. 185.

For dividing the definite integral into two parts

$$y = \int_0^\pi \cos pv \cos(x \sin v) dv, \text{ and } z = \int_0^\pi \sin pv \sin(x \sin v) dv;$$

and using accents to denote differentiation,

$$\begin{aligned} & x^2 y'' + xy' + (x^2 - p^2)y \\ &= \int_0^\pi \{ -x^2 \sin^2 v \cos pv \cos(x \sin v) - x \sin v \cos pv \sin(x \sin v) \\ & \quad + (x^2 - p^2) \cos pv \cos(x \sin v) \} dv \\ &= \{ x \cos v \cos pv \sin(x \sin v) - p \sin pv \cos(x \sin v) \}_0^\pi \\ &= p \sin p\pi; \end{aligned}$$

while

$$\begin{aligned} & x^2 z' + xz' + (x^2 - p^2)z \\ &= \{ -x \cos v \sin pv \cos(x \sin v) + p \cos pv \sin(x \sin v) \}_0^\pi \\ &= -x \sin p\pi; \end{aligned}$$

and therefore y and z each satisfy Bessel's equation if p is an integer; and y or z vanishes as p is odd or even.

The student may verify in a similar manner that if $P_n(\mu)$ (p. 314) satisfies the differential equation of ex. 6, p. 276,

$$\begin{aligned} \frac{d}{d\mu} \left\{ (1 - \mu^2) \frac{dP}{d\mu} \right\} + n(n+1)P &= 0, \\ \pi P_n(\mu) &= \int_0^\pi \{ \mu - \sqrt{(\mu^2 - 1) \cos \phi} \}^n d\phi \\ &= \int_0^\pi \{ \mu + \sqrt{(\mu^2 - 1) \cos \psi} \}^{-n-1} d\psi, \end{aligned}$$

the one integral being transformed into the other by (§ 200)

$$\mu - \sqrt{(\mu^2 - 1) \cos \phi} = \frac{1}{\mu + \sqrt{(\mu^2 - 1) \cos \psi}} = \frac{d\phi}{d\psi}.$$

Putting $\mu = \cos u$, or $\cosh v$, according as μ is $<$ or > 1 ,

$$\frac{d^2 P}{du^2} + \cot u \frac{dP}{du} + n(n+1)P = 0,$$

$$\text{or} \quad \frac{d^2 P}{dv^2} + \coth v \frac{dP}{dv} - n(n+1)P = 0;$$

so that the solution may be written

$$\pi P_n(\cos u) = \int_0^\pi (\cos u - i \sin u \cos \phi)^n d\phi,$$

$$\text{or} \quad \pi P_n(\cosh v) = \int_0^\pi (\cosh v - \sinh v \cos \phi)^n d\phi.$$

Another solution Q_n is obtained by putting

$$Q_n(\mu) = \int_0^\infty \{ \mu + \sqrt{(\mu^2 - 1) \cosh \phi} \}^{-n-1} d\phi,$$

$$\text{or} \quad \int_0^{\coth^{-1} \mu} \{ \mu - \sqrt{(\mu^2 - 1) \cosh \psi} \}^n d\psi,$$

by means of the substitution (§ 200)

$$\{ \mu + \sqrt{(\mu^2 - 1) \cosh \phi} \} \{ \mu - \sqrt{(\mu^2 - 1) \cosh \psi} \} = 1;$$

and $Q_n(\mu) = \infty$ when $\mu = 0$.

Example.—Prove

$$\frac{1}{\cosh v - \cos u} = \sum_{n=0}^{\infty} (2n+1) P_n(\cos u) Q_n(\cosh v).$$

208. Mr. W. M. Hicks has shown (*Phil. Trans.*, 1881) that similar functions are required in the solution of $\nabla^2 V = 0$ (§ 146), in connexion with the anchor rings or *toroidal* surfaces, generated by the revolution of the dipolar circles of § 175.

With the columnar coordinates y, θ, x (§ 132) round Ox as axis, we obtain the transformation

$$-\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{1}{y} \frac{\partial}{\partial y} \left(y \frac{\partial V}{\partial y} \right) + \frac{1}{y^2} \frac{\partial^2 V}{\partial \theta^2};$$

or, on putting $V = \psi y^{-\frac{1}{2}}$,

$$-\nabla^2 V = y^{-\frac{1}{2}} \left\{ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{1}{y^2} \left(\frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{4} \psi \right) \right\};$$

and, on changing from x, y to new variables u, v , conjugate functions given by $u + iv = f(x + iy)$ (§ 133)

$$-\nabla^2 V = J y^{-\frac{1}{2}} \left\{ \frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} + \frac{1}{J y^2} \left(\frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{4} \psi \right) \right\}.$$

With the dipolars of § 175,

$$x + iy = c \tan \frac{1}{2}(u + iv),$$

$$\frac{1}{J} = \left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 = \frac{c^2}{(\cos u + \cosh v)^2} = y^2 \operatorname{cosech}^2 v,$$

$$-\nabla^2 V = y^{-\frac{1}{2}} \left\{ \left(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} \right) \sinh^2 v + \frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{4} \psi \right\}.$$

Then, if ψ varies as $\cos nu \cos m\theta$,

$$-\nabla^2 V = y^{-\frac{1}{2}} \sinh^2 v \left\{ \frac{\partial^2 \psi}{\partial v^2} - n^2 \psi - (m^2 - \frac{1}{4}) \psi \operatorname{cosech}^2 v \right\}.$$

Further, if we put $\psi = \sqrt{(c \sinh v)} P$, $-\nabla^2 V =$

$$\frac{(\cos u + \cosh v)^{\frac{5}{2}}}{c^2} \left(\frac{\partial^2 P}{\partial u^2} + \frac{\partial^2 P}{\partial v^2} + \coth v \frac{\partial P}{\partial v} + \operatorname{cosech}^2 v \frac{\partial^2 P}{\partial \theta^2} + \frac{1}{4} P \right).$$

If we suppose $P = P_n^m \cos nu \cos m\theta$ is a solution of $\nabla^2 V = 0$, then P_n^m , as a function of v only, satisfies the differential equation

$$\frac{d^2 P}{dv^2} + \coth v \frac{dP}{dv} - (n^2 - \frac{1}{4})P - m^2 P \operatorname{cosech}^2 v = 0;$$

and this equation, in the particular case when $m = 0$, is the same as that satisfied by the *zonal harmonic* (p. 276) with $n - \frac{1}{2}$ for n ; so that writing C for $\cosh v$ or μ , and S for $\sinh v$ or $\sqrt{(\mu^2 - 1)}$, the solution is $AP_n + BQ_n$,

$$\text{where } \pi P_n = \int_0^\pi \frac{d\phi}{(C + S \cos \phi)^{n+\frac{1}{2}}}, \quad \pi Q_n = \int_0^\infty \frac{d\phi}{(C + S \cosh \phi)^{n+\frac{1}{2}}}.$$

For, as in ex. 6, v., p. 276,

$$\begin{aligned} S^2 \frac{dP_n}{dC} &= - \int_0^\pi \frac{(n + \frac{1}{2})(S^2 + SC \cos \phi) d\phi}{(C + S \cos \phi)^{n+\frac{3}{2}}} \\ &= (n + \frac{1}{2}) \int_0^\pi \frac{1 - C(C + S \cos \phi)}{(C + S \cos \phi)^{n+\frac{3}{2}}} d\phi \\ &= (n + \frac{1}{2})(P_{n+1} - CP_n); \end{aligned}$$

while taking the form $\pi P_n = \int_0^\pi (C - S \cos \phi)^{n-\frac{1}{2}} d\phi$,

$$S^2 \frac{dP_n}{dC} = (n - \frac{1}{2})(CP_n - P_{n-1});$$

and thence $\frac{d}{dC} \left(S^2 \frac{dP_n}{dC} \right) = (n^2 - \frac{1}{4})P_n$;

and similarly with Q_n .

209. In the general case, when m is any integer, we can verify that the solution of

$$\frac{d}{dC} \left(S^2 \frac{dP}{dC} \right) - (n^2 - \frac{1}{4})P - \frac{m^2 P}{S^2} = 0$$

is

$$P = AP_n^m + BQ_n^m,$$

where

$$\pi P_n^m = \int_0^\pi \frac{\cos m\phi d\phi}{(C + S \cos \phi)^{n+\frac{1}{2}}}, \quad \pi Q_n^m = \int_0^\infty \frac{\cosh m\phi d\phi}{(C + S \cosh \phi)^{n+\frac{1}{2}}}.$$

For now, taking Q_n^m , we prove that

$$\begin{aligned} & \left(S^2 \frac{d}{dC} - mC \right) Q_n^m + (n - m - \tfrac{1}{2}) S Q_n^{m+1} \\ &= - \left\{ \frac{S \sinh(m+1)\phi + C \sinh m\phi}{\pi(C + S \cosh \phi)^{n+\frac{1}{2}}} \right\}_0^\infty = 0, \text{ if } m < n; \end{aligned}$$

and thus, as in § 87,

$$\begin{aligned} S^m \frac{d^m Q_n}{dC^m} &= S \frac{d}{dC} \left(S \frac{d}{dC} - \frac{C}{S} \right) \left(S \frac{d}{dC} - 2 \frac{C}{S} \right) \dots \left(S \frac{d}{dC} - m - 1 \cdot \frac{C}{S} \right) Q_n \\ &= (-1)^m (n - \tfrac{1}{2})(n - \tfrac{3}{2}) \dots (n - m - \tfrac{1}{2}) Q_n^m. \end{aligned}$$

Similarly

$$\begin{aligned} & \left(S^2 \frac{d}{dC} + mC \right) Q_n^m + (n + m - \tfrac{1}{2}) S Q_n^{m-1} \\ &= \left\{ \frac{S \sinh(m-1)\phi + C \sinh m\phi}{\pi(C + S \cosh \phi)^{n+\frac{1}{2}}} \right\}_0^\infty = 0; \end{aligned}$$

and thence $\frac{d}{dC} \left(S^2 \frac{d}{dC} Q_n^m \right)$

$$\begin{aligned} &= -m Q_n^m - mC \frac{d}{dC} Q_n^m - (n + m - \tfrac{1}{2}) \left(\frac{C}{S} Q_n^{m-1} + S \frac{d}{dC} Q_n^{m-1} \right) \\ &= \left(n^2 - \tfrac{1}{4} + \frac{m^2}{S^2} \right) Q_n^m, \end{aligned}$$

thus verifying the differential equation; and the procedure is the same with P_n^m .

The substitution $\tan \tfrac{1}{2}\theta = \frac{\sqrt{(C + S \cosh \phi)}}{\cosh \tfrac{1}{2}u \sinh \phi}$ will give

$$\int_0^\pi \frac{d\theta}{(C - \cos \theta)^{\frac{1}{2}}} = \int_0^\infty \frac{\sqrt{2} d\phi}{(C + S \cosh \phi)^{\frac{1}{2}}} = \sqrt{2} \pi Q_0;$$

and generally if we denote $\int_0^\pi \frac{\cos n\theta d\theta}{(C - \cos \theta)^{\frac{1}{2}}}$ by πR_n , then

$$S^2 \frac{dR_n}{dC} - (n + \frac{1}{2})(R_{n+1} - CR_n) = \left\{ \frac{C \sin n\theta - \sin(n+1)\theta}{(C - \cos \theta)} \right\}_0^\pi = 0,$$

$$S^2 \frac{dR_n}{dC} - (n - \frac{1}{2})(CR_n - R_{n-1}) = \left\{ \frac{-C \sin n\theta + \sin(n-1)\theta}{(C - \cos \theta)^{\frac{1}{2}}} \right\}_0^\pi = 0,$$

n being an integer; so that R_n satisfies the same equations as Q_n ; and since $R_0 = \sqrt{2}Q_0$, therefore $R_n = \sqrt{2}Q_n$.

By differentiation with respect to C ,

$$\begin{aligned} & \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \dots (m - \frac{1}{2}) \int_0^\pi \frac{S^m \cos n\theta d\theta}{(C - \cos \theta)^{m+\frac{1}{2}}} \\ &= \pi(-1)^m S^m \frac{d^m R_n}{dC^m} = \pi \sqrt{2}(-1)^m S^m \frac{d^m Q_n}{dC^m} \\ &= \pi \sqrt{2} (n - \frac{1}{2})(n - \frac{3}{2}) \dots (n - m - \frac{1}{2}) Q_n^m \\ &= \sqrt{2} (n - \frac{1}{2})(n - \frac{3}{2}) \dots (n - m - \frac{1}{2}) \int_0^\infty \frac{\cosh m\phi d\phi}{(C + S \cosh \phi)^{n+\frac{1}{2}}}. \end{aligned}$$

When the dipolar system of § 175 is revolved round Oy , we shall find in the same manner, on putting

$$V = \psi x^{-\frac{1}{2}}, \text{ and } \psi = \sqrt{(c \sin u)} P,$$

$$\begin{aligned} -\nabla^2 V &= x^{-\frac{5}{2}} \left\{ \left(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} \right) \sin^2 u + \frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{4} \psi \right\} \\ &= \frac{(\cos u + \cosh v)^{\frac{5}{2}}}{c^2} \left(\frac{\partial^2 P}{\partial u^2} + \cot u \frac{\partial P}{\partial u} + \frac{\partial^2 P}{\partial v^2} + \operatorname{cosec}^2 u \frac{\partial^2 P}{\partial \theta^2} - \frac{1}{4} P \right); \end{aligned}$$

and now if $P = P_n^m \cos nv \cos m\theta$ is a solution of $\nabla^2 V = 0$, then P_n^m , as a function of u only, satisfies the equation

$$\frac{d^2 P}{du^2} + \cot u \frac{dP}{du} - (n^2 + \frac{1}{4})P - m^2 P \operatorname{cosec}^2 u = 0,$$

of which the solution is obtained from the preceding results by replacing $\cosh v$ and $\sinh v$ by $\cos u$ and $\sin u$.

In the particular case of $m = 0$, the order of the corresponding zonal harmonics would be $-\frac{1}{2} + in$, and therefore imaginary; these are called *Mehler functions*.

210. *Lagrange's Theorem.*

We can establish Lagrange's Theorem (§ 150) by the rule for the differentiation of a definite integral; for, if

$$y = a + x\phi y,$$

$$\frac{1}{n!} \frac{\partial}{\partial a} \int_a^y (x\phi z + a - z)^n f'z dz$$

$$= \frac{1}{(n-1)!} \int_a^y (x\phi z + a - z)^{n-1} f'z - \frac{x^n}{n!} (\phi a)^n f'a;$$

so that differentiating again $n-1$ times with respect to a , and denoting

$$\frac{1}{n!} \frac{\partial^n}{\partial a^n} \int_a^y (x\phi z + a - z)^n f'z dz \text{ by } R_n,$$

we find that

$$R_n = R_{n-1} - \frac{x^n}{n!} \frac{\partial^{n-1}}{\partial a^{n-1}} \{ (\phi a)^n f'a \},$$

while $R_0 = fy - fa$; so that

$$R_n = fy - fa - \sum_{r=1}^{r=n} \frac{x^r}{r!} \frac{\partial^{r-1}}{\partial a^{r-1}} \{ (\phi a)^r f'a \},$$

which is Lagrange's Theorem.

211. *Integration considered as Summation.*

Hitherto we have considered Integration as an operation to be performed by the reversion of Differentiation; but now, by the aid of Taylor's Theorem, we can exhibit it in its fundamental conception as the Summation of a number of infinitesimal elements, and show that the result is the same as before.

For let any function fx , between the fixed values a and b of x , be considered for the consecutive infinitesimally graduated sequence of values $a, x_1, x_2, \dots x_r, \dots x_n, b$, of the variable x .

Then by Taylor's Theorem (§§ 114, 118), using the symbol Δx_r to denote the difference $x_{r+1} - x_r$,

$$fx_1 - fa - \Delta a f'a = \frac{1}{2} \Delta a^2 f''(a + \theta \Delta a),$$

$$fx_2 - fx_1 - \Delta x_1 f'x_1 = \frac{1}{2} \Delta x_1^2 f''(x_1 + \theta_1 \Delta x_1),$$

$$\dots\dots\dots$$

$$fx_{r+1} - fx_r - \Delta x_r f'x_r = \frac{1}{2} \Delta x_r^2 f''(x_r + \theta_r \Delta x_r),$$

$$\dots\dots\dots$$

$$fb - fx_n - \Delta x_n f'x_n = \frac{1}{2} \Delta x_n^2 f''(x_n + \theta_n \Delta x_n).$$

Adding these equations, and using the symbol Σ to denote summation for all integral values of r from 1 to n ,

$$fb - fa - \Sigma \Delta x_r f'x_r = \Sigma \frac{1}{2} \Delta x_r^2 f''(x_r + \theta_r \Delta x_r).$$

Now if o denotes the greatest of all the evanescent quantities, such as $\Delta x_r f''(x_r + \theta_r \Delta x_r)$, then

$fb - fa - \Sigma f'x_r \Delta x_r$ is less than $\frac{1}{2} o \Sigma (x_{r+1} - x_r)$ or $\frac{1}{2} o(b - a)$; and this quantity ultimately vanishes; so that replacing Σ and Δ by \int and d , as in § 44,

$$fb - fa - \int_a^b f'x dx = 0,$$

which establishes the fact that Integration considered as Summation is a process the reverse of Differentiation.

For instance in Quadrature (§ 41), we determine the arithmetic mean of the ordinates of a curve, and call this the mean ordinate; and now the area between any two ordinates is a rectangle of the same breadth, and height equal to the mean ordinate.

Suppose however that M , the geometric mean of the equispaced ordinates of the curve $y = fx$ between $x = a$ and $x = b$ is required; then $M = \sqrt[n]{(fx_1 \cdot fx_2 \dots fx_r \dots fx_n)}$,

and $\log M = \log \sqrt[n]{\Sigma \log fx} = \log \Sigma \log fx \Delta x / (b - a)$

$$= \int_a^b \log fx dx / (b - a),$$

so that
$$M = \exp \left\{ \int_a^b \log f(x) dx / (b-a) \right\}.$$

As an application, prove that the geometric mean of the ordinates of the curve $y = \sin x$, from 0 to π is $\frac{1}{2}$.

212. Approximate Quadrature.

Various rules are employed in practice for determining the area of an irregular figure, when a Planimeter is not used.

The simplest is that usually employed with an Indicator Diagram; the breadth of the closed contour is measured at a number of equidistant ordinates which divide the area into a number n of strips, and the arithmetic mean of these is taken as the mean breadth; more strictly we take

$$A = l(\frac{1}{2}y_0 + y_1 + y_2 + \dots + y_{n-1} + \frac{1}{2}y_n),$$

where the y 's denote the breadth of the figure at the different ordinates, and l denotes the spacing of the ordinates; this is equivalent to replacing the area by trapezoids, bounded by the same ordinates.

In Simpson's rule a higher approximation is obtained by taking the curve through the tops of three equidistant ordinates as a parabola with axis parallel to the ordinates; and now if y_{m-1} , y_m , y_{m+1} denote three consecutive ordinates and l their spacing, the parabola is taken as given by
$$y = a + bx + \frac{1}{2}cx^2;$$
 and if the origin is taken at the foot of the middle ordinate, as in § 77,

$$a = y_m, \quad 2bl = y_{m+1} - y_{m-1}, \quad cl^2 = y_{m+1} - 2y_m + y_{m-1}.$$

The area of the two strips is now given by

$$\int_{-l}^l y dx = 2al + \frac{1}{3}cl^3 = \frac{1}{3}l(y_{m+1} + 4y_m + y_{m-1});$$

and
$$A = \frac{1}{3}l(y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 4y_{2n} + y_{2n+1}),$$
 which is called *Simpson's Rule*.

We may replace the boundary by the curve of the third degree

$$y = a + bx + \frac{1}{2}cx^2 + \frac{1}{6}dx^3,$$

passing through the tops of four equidistant ordinates $y_m, y_{m+1}, y_{m+2}, y_{m+3}$; and now the area of the three strips

$$\begin{aligned} \int_0^{3l} y dx &= 3l(a + \frac{3}{2}bl + \frac{3}{2}cl^2 + \frac{9}{8}el^3) \\ &= \frac{3}{8}l(y_m + 3y_{m+1} + 3y_{m+2} + y_{m+3}). \end{aligned}$$

It is convenient to take an odd number of ordinates and an even number of strips, and a general formula for approximate quadrature, with $2n+1$ ordinates and $2n$ strips, has been given by Newton.

With 7 ordinates and 6 strips, the expression for the area is, very approximately,

$$A = \frac{3}{10}l\{6u_m + u_{m-1} + u_{m+1} + 5(u_{m-2} + u_{m+2}) + u_{m-3} + u_{m+3}\},$$

called *Weddle's Rule*.

In these expressions for the area, the y 's may represent breadths as well as ordinates; or they may represent the areas of parallel equidistant sections of a solid body; for instance, horizontal and vertical sections of a ship.

Simpson's Rule may also be used for finding the centroid and moment of inertia of the area, by putting

$$\begin{aligned} A\bar{x} &= \int xy dx = \frac{1}{3}l(x_0y_0 + 4x_1y_1 + 2x_2y_2 + \dots), \\ A\bar{y} &= \int \frac{1}{2}y^2 dx = \frac{1}{6}l(y_0^2 + 4y_1^2 + 2y_2^2 + \dots); \\ Ak_x^2 &= \int \frac{1}{3}y^3 dx = \frac{1}{9}l(y_0^3 + 4y_1^3 + 2y_2^3 + \dots), \\ Ak_y^2 &= \int x^2 y dx = \frac{1}{3}l(x_0^2y_0 + 4x_1^2y_1 + 2x_2^2y_2 + \dots). \end{aligned}$$

As a numerical application, calculate these quantities for the horizontal section of a ship, given the half ordinates from the median line as .5, 6, 10, 12, 12.4, 12.5, 12.5, 12.5, 12.4, 12.3, 11, 8, and .5 feet; the ordinates being spaced at 12 feet.

213. *Differential Equations.*

In Chapters III. and V. the principles of Differentiation have been illustrated in the formation of certain important *Differential Equations* from their primitive relations; but now it is requisite to reverse this process, and to discover the primitive relation or solution where possible, when the differential equation is given.

The complete subject of Differential Equations would soon lead us beyond the scope of the present work, and the reader is referred to the treatises of Forsyth and Woolsey Johnson; at the same time, however, it is possible to indicate the procedure in a few simple cases, often required in Dynamics and Electricity.

The simplest case of frequent occurrence is the *linear differential equation with constant coefficients*, of the

$$\text{form} \quad \frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + A_{n-1} \frac{dy}{dx} + A_n y = V,$$

where the A 's are constant, and V is a given function of x .

Using D to denote the operation d/dx , we may write the equation $F(D)y = V, \dots \dots \dots (1).$

where $F(D)$ is a rational integral algebraical function of D ; and now the solution will consist of two parts, one of them being any *particular solution* of this equation (1), while the other, called the *complementary function*, is the general solution of the equation

$$F(D)y = 0. \dots \dots \dots (2).$$

We have shown in § 68 that $F(D)e^{bx} = e^{bx}Fb$, so that $y = e^{bx}$ will be a solution of (2) if $Fb = 0$; and if the n roots b_1, b_2, b_3, \dots of this equation $Fb = 0$ are all real and distinct, the complementary function will be

$$y = B_1 e^{b_1 x} + B_2 e^{b_2 x} + \dots + B_n e^{b_n x},$$

where B_1, B_2, \dots, B_n are n arbitrary constants.

If m roots are equal to b , the corresponding terms in the complementary function are, as in ex. 3, p. 143,

$$(C_0 + C_1x + \dots + C_{m-1}x^{m-1})e^{bx}.$$

If a pair of roots are conjugate imaginary, $\alpha \pm i\beta$, the corresponding terms, by De Moivre's theorem, are written

$$e^{ax}(A \cos \beta x + B \sin \beta x);$$

and if these imaginary roots are repeated m times, the corresponding terms are

$$e^{ax} \sum_{s=0}^{s=m-1} x^s (A_s \cos \beta x + B_s \sin \beta x).$$

In determining the *particular solution* of (1), we write the equation symbolically

$$y = (FD)^{-1}V,$$

where $(FD)^{-1}$ can be resolved into a series of partial fractions of the form $A(D-b)^{-1}$, or $B(D-b)^{-r}$.

Consider the separate terms of V , of the form $x^n, e^{ax}, e^{bx}, \sin mx, \dots$

(i.) $(FD)^{-1}x^n$ is found by expanding $(FD)^{-1}$ in ascending powers of D as far as D^n , and performing the differentiations; terms beyond D^n may be neglected, since $D^{n+1}x^n = 0, \dots$

$$(ii.) \quad (FD)^{-1}e^{ax} = e^{ax}(Fa)^{-1},$$

and $(FD)^{-1}e^{ax}fx = e^{ax}\{F(D+a)\}^{-1}fx$, (§ 70).

(iii.) Thus, if $Fb = 0$,

$$(D-b)^{-r}e^{bx} = e^{bx}D^{-r}. 1 = e^{bx}\left(C_0 + C_1x + \dots + C_{r-1}x^{r-1} + \frac{x^r}{r!}\right),$$

since D^{-r} represents the operation of integration, repeated r times.

$$(iv.) \quad (D-b)^{-r}\cos mx = (D+b)^r(D^2 - b^2)^{-r}\cos mx \\ = (D+b)^r(-m^2 - b^2)^{-r}\cos mx,$$

(§ 68); and so on for $(D^2 + m^2)^{-r}\cos mx, \dots$, by employing the exponential values of $\cos mx, \dots$ (§ 111).

Consider for instance the differential equation

$$\frac{d^2x}{dt^2} + 2n \cos \beta \frac{dx}{dt} + n^2x = E \cos pt,$$

for the *forced vibrations*, due to $E \cos pt$, of a system of which the natural motion is given by the complementary

function $x = ae^{-nt \cos \beta} \cos (nt \sin \beta + \epsilon)$

(ex. 6, p. 139), representing S.H.M. (§ 103) of period $2\pi/(n \sin \beta)$, with a *modulus of decay* $n \cos \beta$.

The particular solution is

$$\begin{aligned} x &= \frac{E \cos pt}{D^2 + 2n \cos \beta \cdot D + n^2} \\ &= \frac{D^2 - 2n \cos \beta \cdot D + n^2}{(D^2 + n^2)^2 - 4n^2 \cos^2 \beta \cdot D^2} E \cos pt \\ &= \frac{(n^2 - p^2) E \cos pt + 2np \cos \beta \cdot E \sin pt}{(n^2 - p^2)^2 + 4n^2 p^2 \cos^2 \beta} \\ &= \frac{E \cos[pt - \tan^{-1}\{2np \cos \beta / (n^2 - p^2)\}]}{\sqrt{(n^4 + 2n^2 p^2 \cos 2\beta + p^4)}}, \end{aligned}$$

representing a S.H.M. of amplitude

$$E / \sqrt{(n^4 + 2n^2 p^2 \cos 2\beta + p^4)},$$

and change of *phase* $\tan^{-1}\{2np \cos \beta / (n^2 - p^2)\}$.

Examples.—Solve the differential equations

- (i.) $y'' - 5y' + 6y = x^2 + e^{4x} + \sin 4x + xe^{2x} + x^2e^{3x}$,
- (ii.) $y'' + 4y = \sin x + \sin 2x$,
- (iii.) $y'' - 6y' + 9y = \cosh 3x + x \cos 2x$,
- (iv.) $y'' + 2y' + 5y = e^x \sin 2x + \cosh x \cos 2x$,
- (v.) $y''' + y'' - y' - y = xe^x + \cosh x$,
- (vi.) $y''' - 8y'' + 16y' = \sinh 2x + \cosh 4x$,
- (vii.) $y''' + y' + 10y = \cosh 2x + \sinh x \sin 2x$,
- (viii.) $y''' - 26y' - 60y = \sinh 6x + (A + B \cosh 3x) \cos x$,
- (ix.) $y''' + 2y'' + x = \sin 2x + \sin x$,
- (x.) $y''' - 2y'' + 3y' - 2y = x + e^{-\frac{1}{2}x} (A + B \cos \frac{1}{2}\sqrt{3}x)$,
- (xi.) $y''' + 32y' + 48y = x \sinh 2x + \cosh 2x \cos (2\sqrt{2}x)$.

214. With variable coefficients P, Q, R, \dots , given functions of x , the general linear differential equation of the first order is written $y' + Py = Q$.

To solve this, we can proceed as in § 88, and put $y = uv$, when
 $u'v + u(v' + Pv) = Q$;
 and now if v is chosen so that

$$v' + Pv = 0, \text{ or } v = \exp(-\int P dx),$$

then $u' = Q \exp \int P dx$,

and $u = \int Q(\exp \int P dx) dx + C$,

$$y = uv = \exp(-\int P dx) \left\{ \int Q(\exp \int P dx) dx + C \right\}.$$

Thus, for instance, the solution of the equation

$$L \frac{di}{dt} + Ri = E \cos pt,$$

connecting R the resistance in *ohms*, i the current in *ampères*, $E \cos pt$ the variable electromotive force of a dynamo in *volts*, and $L di/dt$ the counter electromotive force of induction, so that L is the inductance in *secohms* (Fleming, *The Alternate Current Transformer*) is

$$\begin{aligned} i &= e^{-Rt/L} \left(\int E \cos pt \cdot e^{Rt/L} dt + C \right) \\ &= \frac{RE \cos pt + pLE \sin pt}{R^2 + p^2 L^2} + C e^{-Rt/L} \\ &= \frac{E \cos \{pt - \tan^{-1}(pL/R)\}}{\sqrt{(R^2 + p^2 L^2)}} + C e^{-Rt/L}. \end{aligned}$$

But considered as a differential equation with constant coefficients, $C e^{-Rt/L}$ is the complementary function, while the particular solution is, as before,

$$\begin{aligned} i &= \frac{E \cos pt}{LD + R} = \frac{R - LD}{R^2 - L^2 D^2} E \cos pt \\ &= \frac{RE \cos pt + pLE \sin pt}{R^2 + p^2 L^2} = \frac{E \cos \{pt - \tan^{-1}(pL/R)\}}{\sqrt{(R^2 + p^2 L^2)}}. \end{aligned}$$

215. The general linear differential equation of the second order (§ 88) $y'' + 2Py' + Qy = R$,

on putting $y = uv$, and choosing v so that

$$v' + Pv = 0,$$

becomes

$$u'' + Iu = S,$$

where $I = Q - P^2 - P'$, $S = R/v$;

and now, if we can determine the complementary function $Au_1 + Bu_2$, then, as in § 88,

$$u_1'u_2 - u_1u_2' = C;$$

and the particular solution is

$$Cu = u_1 \int Su_2 dx - u_2 \int Su_1 dx.$$

Thus, as on pp. 183-185, Bessel functions are required in the solution of Riccati's equation, in which $I = kx^m$; and the student can easily verify that

$$u = A x \int_0^{\frac{1}{2}\pi} \frac{\cos}{\cosh} (mx^n \sin \theta) (\cos \theta)^{\frac{1}{n}} d\theta \\ + B \int_0^{\infty} \frac{\cosh}{\cos} (mx^n \sinh \phi) (\cosh \phi)^{-\frac{1}{n}} d\phi$$

is the solution when $I = \pm m^2 n^2 x^{2n-2}$, with certain restrictions on the value of n .

Again, *zonal harmonics* (pp. 276, 314, 431) give the solution when

$$I = (n + \frac{1}{2})^2 + \frac{1}{4} \operatorname{cosec}^2 u, \text{ or } -(n + \frac{1}{2})^2 + \frac{1}{4} \operatorname{cosech}^2 v;$$

and the *toroidal* and *Mehler functions* of §§ 208, 209, when

$$I = -n^2 - (m^2 - \frac{1}{4}) \operatorname{cosech}^2 v, \text{ or } -n^2 - (m^2 - \frac{1}{4}) \operatorname{cosec}^2 u.$$

But no general method has yet been discovered for the solution of the linear differential equation of the second order in the canonical form

$$u'' + Iu = 0,$$

where I is an arbitrary function of x ; nor *a fortiori* for differential equations of a higher order.

216. *Simultaneous differential equations* with constant coefficients, as those given in exs. 3 and 4, p. 219, may be solved by assuming $x = Pe^{bt}$, $y = Qe^{bt}$, or $x = P \sin(nt + \epsilon)$, $y = Q \sin(nt + \epsilon')$, and eliminating P/Q , and ϵ, ϵ' , when an equation for the determination of b or n will be obtained.

Thus for instance, in the theory of the Induction Coil, the differential equations

$$L \frac{di_1}{dt} + M \frac{di_2}{dt} + Ri_1 = E \cos pt,$$

$$M \frac{di_1}{dt} + N \frac{di_2}{dt} + Si_2 = 0,$$

for the primary current i_1 , and the secondary current i_2 , are solved by assuming

$$i_1 = I_1 \cos(pt + \epsilon_1), \quad i_2 = I_2 \cos(pt + \epsilon_2),$$

and determining $I_1, I_2, \epsilon_1, \epsilon_2$.

The formation of *partial* differential equations has been illustrated in Chap. V., from which the solution of simple partial differential equations of the first order may be inferred; but for further developments the reader must consult the Treatises on Differential Equations mentioned above.

APPENDIX I.

Figure 58 given here is intended to serve as a guide to the student in plotting the curves given on p. 24, and elsewhere; the curves are the graphs of the equations of ex. 1, p. 24.

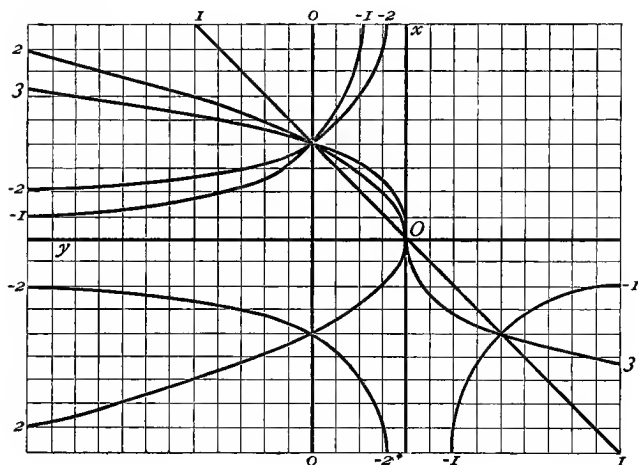


Fig.58

APPENDIX II.

On p. 344, the theorems of *equimomental lines*, and of the *equimomental ellipses* which they touch, have been introduced; and the reader is referred to Routh's *Rigid Dynamics*, Chap. I., or Clifford's *Dynamic*, vol. II., for a demonstration of these theorems.

APPENDIX III.

The following Table gives the value of

$$u = \int_0^{\theta} \sec \theta d\theta = \log(\sec \theta + \tan \theta)$$

for every degree of an angle whose c.m. is θ ; and then

$$\begin{aligned} \cosh u &= \sec \theta, \quad \sinh u = \tan \theta, \\ \tanh u &= \sin \theta, \quad \tanh \frac{1}{2}u = \tan \frac{1}{2}\theta. \end{aligned}$$

For values of u greater than about 4 the Table fails; but then it is generally sufficiently accurate to take

$$\cosh u = \sinh u = \frac{1}{2}e^u,$$

neglecting e^{-u} ; so that, M denoting the modulus $\log_{10}e$,

$$\log_{10} \cosh u = Mu - \log 2.$$

To a closer approximation

$$\log_{10} \cosh u = Mu - \log 2 + Me^{-2u} \dots$$

$$\log_{10} \sinh u = Mu - \log 2 - Me^{-2u} \dots$$

$$\log_{10} \tanh u = -2Me^{-2u} \dots$$

(*Proposed Tables of Hyperbolic Functions*; Report to the British Association 1888, by Prof. Alfred Lodge).

This is the Table devised originally by Edward Wright, 1599, and called by him a *Table of Meridional Parts*; by means of which he measured the latitude u in Mercator's chart, corresponding to the angular latitude θ on the globe (§ 176); previously to Wright's theory, the degrees of latitude had been laid off empirically.

θ	u	θ	u	θ	u
°		°		°	
0	0.00000	30	0.54931	60	1.31696
1	0.01745	31	0.56956	61	1.35240
2	0.03491	32	0.59003	62	1.38899
3	0.05238	33	0.61073	63	1.42679
4	0.06987	34	0.63166	64	1.46591
5	0.08738	35	0.65284	65	1.50685
6	0.10491	36	0.67428	66	1.54855
7	0.12248	37	0.69599	67	1.59232
8	0.14008	38	0.71699	68	1.63794
9	0.15773	39	0.74029	69	1.68557
10	0.17543	40	0.76291	70	1.73542
11	0.19318	41	0.78586	71	1.78771
12	0.21099	42	0.80917	72	1.84273
13	0.22886	43	0.83284	73	1.90079
14	0.24681	44	0.85680	74	1.96226
15	0.26484	45	0.88137	75	2.02759
16	0.28295	46	0.90628	76	2.09732
17	0.30116	47	0.93163	77	2.17212
18	0.31946	48	0.95747	78	2.25280
19	0.33786	49	0.98381	79	2.34040
20	0.35638	50	1.01068	80	2.43625
21	0.37501	51	1.03812	81	2.54209
22	0.39377	52	1.06616	82	2.66031
23	0.41266	53	1.09483	83	2.79422
24	0.43169	54	1.12418	84	2.94870
25	0.45088	55	1.15423	85	3.13130
26	0.47021	56	1.18505	86	3.35467
27	0.48872	57	1.21667	87	3.64253
28	0.50939	58	1.24916	88	4.04813
29	0.52925	59	1.28257	89	4.74135
30	0.54931	60	1.31696	90	infinite

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